Growth, Income Distribution and Policy Implications of Automation*

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Abstract

Motivated by recent discussions about the distributional implications of automation and the possibility of a "universal basic income" (UBI) program to address them, we analyze the distributional and political economy implications of automation in a task-based model of production. We conceptualize automation as a shift in the relative productivity of capital at certain tasks that reduces the set of tasks done by labor. We capture distributional concerns by including two kinds of agents — workers, who supply labor, and entrepreneurs, who own capital. After characterizing the equilibrium with fixed tax rates, we endogenize policy variables by computing the political economy equilibrium under majority voting, where policymakers have access to a capital tax and a transfer to workers (a "universal basic income"). We quantitatively study an episode of automation in a calibrated model, and find that workers prefer higher capital taxes in the long run, but lower capital taxes during the transition. Interestingly, this results in larger welfare gains for entrepreneurs than for workers relative to a constant tax regime, suggesting that capital owners stand to benefit from the institution of a UBI in a policy regime that maximizes worker welfare. In other results, we derive conditions under which the economy experiences full automation, characterize comparative statics of traditional technical progress vs. automation, give a condition under which automation may lower long-term wages, and show that all kinds of capital should be taxed at the same rate (i.e. no "robot taxes").

Keywords: Automation, labor share, economic growth, majority voting

JEL Classification: E32; F34; F41; F44; F47; H63

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1 Introduction

Technical progress has been the driving force of economic growth since the dawn of agriculture, and particularly since the Industrial Revolution ushered in the era of modern growth. This economic growth has been accompanied by the replacement of human labor, first by animals and then by machines. While this process has doubtless harmed some individual workers, historically the overall effect has been to boost the incomes of workers and owners of capital in tandem, as evidenced by the roughly constant share of labor in national income over time. In recent years, however, there has been growing concern that the current episode of technical progress (which we call automation) is qualitatively different, with much more drastic implications for growth and the income distribution. As this automation is substituting for human labor in a broader manner than previous episodes of technical change (which we call traditional technical progress), it is likely to both accelerate economic growth (by replacing scarce labor with reproducible capital), and decrease the labor share of national income. Since most households derive most of their income from labor, whereas capital ownership is concentrated, this will likely lead to growing inequality and political instability. Some experts, such as Brynjolfsson and McAfee (2014) and Ford (2015), forecast these trends to continue and even accelerate in the coming years.

Public discussion of these trends has included consideration of a wide range of fiscal policy tools to ameliorate these effects. Some commentators have called for an expansion of safety net and income assistance programs, or even for the institution of widespread unconditional monetary transfers to all citizens, sometimes called a "universal basic income" (UBI).[1] The idea of UBI, the most novel of the policies being discussed, has garnered increasing public support, rising from 12 percent ten years ago to 48 percent in 2018 in a poll of the American public CNBC (2018). Moreover, governments throughout the world have started piloting projects to understand how individuals respond to such a public policy tool and its broader economic consequences Forbes (2015); Wired (2017).

Inspired by this discussion, we construct a model of automation in which distributional considerations and policy responses are front and center. We analyze an episode of automation in a task-based model of production, which has recently become standard in the literature on automation.[2] Rather than adopting a representative agent framework, we include workers and entrepreneurs as separate agents. After describing the steady state and

[1]While the idea of unconditional cash transfers has come to the fore recently, it is an idea with a long history in economics. Milton Friedman famously advocated replacing our current system of means-tested welfare programs and progressive income taxation with a "negative income tax" system— a scheme whereby households would pay a flat tax rate, and would receive an unconditional transfer from the governmentFriedman (1966).

[2]For example, Aghion et al. (2017) and Acemoglu and Restrepo (2018c) use a similar model of production.

comparative statics, we analyze the political economy equilibrium of the model under majority voting, in which policymakers have access to a set of transfers which can implement a "universal basic income". This allows us to speak directly to the current debate, in which distributional concerns are foremost and universal transfer programs are being contemplated.

Using this model, we quantitatively analyze an episode of automation, contrasting the results under majority voting with those under fixed tax rates. Unsurprisingly, workers vote for lower labor taxes and somewhat higher capital taxes in the long run. Perhaps surprisingly, we find that workers vote for lower capital taxes during the transition, because they benefit from faster accumulation of capital by entrepreneurs, which causes their wages and UBI transfers to grow as well. Interestingly, the welfare gains to entrepreneurs from this policy regime are much greater than the gains to workers.[3] This result is particularly striking because this policy is set without consideration of entrepreneur welfare, since workers are assumed to be in the majority. While this may be surprising, it arises from a simple and intuitive reason: optimal policy requires accelerated accumulation of capital, which is implemented through lower capital taxes along the transition. The resulting benefit for workers in terms of faster wage growth is offset by reduced transfers in the interim, whereas for entrepreneurs the lower capital taxes simply move consumption forward in time (in tandem with capital accumulation) without any offsetting forces being operational. Thus the welfare gain from adopting optimal policy is first-order for entrepreneurs, but second-order for workers. This difference may underlie interest in UBI by many tech entrepreneurs, since when workers are able to receive a direct share of the income from automation, they support policies that benefit entrepreneurs much more than themselves.

We also obtain a few other interesting results. As has previously been found in similar models, it is possible that a steady state of the model does not exist. This occurs when the share of tasks done by capital approaches 1, so that continuous growth is possible through the accumulation of capital alone. We give a simple and intuitive condition for when this occurs, both for fixed taxes, and again in our political economy model. Our comparative statics find that automation necessarily lowers the labor share. While in our standard model automation necessarily raises wages in the long run, we consider an extension in which the opposite can occur, because the tasks being automated have higher labor productivity than remaining tasks. Finally, we examine whether workers will ever choose to tax different types of capital at different rates (i.e. institute "robot taxes" aimed at automation directly), and find that they will never choose to do so.

[3]For example, in our baseline calibration with low task substitutability, the welfare gain to workers from an episode of automation is 26.1% (in consumption-equivalent terms) under majority voting compared with 24.6% for fixed policy, whereas for entrepreneurs the equivalent numbers are 84.5% and 57.7%.

Intuitively, and in line with past work, we find that an important parameter to the consequences of automation is the elasticity of substitution between tasks, which we call σ. Since in equilibrium, one set of tasks will be done by capital and a different set by labor, σ is closely related to the elasticity of substitution between capital and labor. When tasks are gross substitutes ($\sigma > 1$), traditional technical progress can lead to sustained economic growth without complete automation, and the labor share decreases. By contrast, when task substitutability is low ($\sigma < 1$), full automation (defined as strictly positive capital productivity for all tasks) is a precondition for sustained economic growth, and the labor share increases under traditional technical progress. These results suggest that automation has larger distributional consequences with task substitutability is low, which accords with the results of our quantitative analysis. For example, in our calibrated model with $\sigma = 0.8$, an episode of automation increases entrepreneur welfare by 24.6% and worker welfare by 57.7% (in consumption equivalent terms), whereas when $\sigma = 1.2$ an analogous episode of automation increases entrepreneur welfare by 29.4% and worker welfare by 49%.[4]

Related Literature. We use task-based modeling of production and model automation as a process of replacing human labor with capital in particular tasks. This framework is common in the theoretical literature on automation, being used in e.g. Aghion et al. (2017) and several recent papers by Acemoglu and Restreppo (chiefly Acemoglu and Restrepo (2018c), Acemoglu and Restrepo (2018a), and Acemoglu and Restrepo (2018b)). We find this modeling of automation very intuitive: the basic dynamics of substituting capital for labor due to automation operates at the level of the tasks that are undertaken for the production of goods. Moreover, with this modeling of automation in view, the task-based representation of production becomes a natural choice because the traditional division of all the tasks performed in the productive sector into distinct goods is neither central nor necessary. Furthermore, disregarding good-based labeling of tasks and instead focusing on their factor-usage allows a sharper analysis of the questions that interest the literature on automation. Our contribution relative to the existing theoretical literature on automation comes from our focus on distributional consequences, and our political economy and quantitative analysis.

The task-based model of production and automation has been used to explore a number of important issues related to automation. For example, Acemoglu and Restrepo (2020) apply the task-based framework in a structural model to examine the causes of the recent decline in manufacturing employment. They find that a significant share has been due to automation with a more limited role played by globalization.

[4]These numbers are the percentage increase in consumption in the initial steady state (with labor supply fixed at its initial level) that would make entrepreneurs and workers respectively indifferent to an episode of automation that doubles steady state output, which we analyze in Section 5.1.

Despite the abundance of empirical evidence on the role of automation in income distribution, and the likely importance of a continuation of these trends, there has been relatively little theoretical analysis of the distributional consequences of automation, particularly in a political economy framework. A few papers have examined the consequences of automation within the framework of standard growth models. In particular, our work vis-a-vis growth and income distribution implications is related to Aghion et al. (2017), who consider how automation and artificial intelligence (AI) alter standard results in models of economic growth. An early example is Zeira (1998), who argues that different incentives to engage in automation may play a role in explaining international differences in productivity.

The specific model that we use bears a significant similarity to that in Acemoglu and Restrepo (2018c), so it is worth describing in detail the differences between our analysis and theirs. They have a representative agent, while we have workers and entrepreneurs as separate agents, allowing us to speak directly to distributional concerns. We consider variable taxes and transfers, and endogenize policy using a political economy model, while they do not focus on policy at all. They instead focus on long-term balanced growth, endogenizing the process of automation and giving theoretical results, whereas we take an episode of automation to be exogenously given and analyze the policy response, including a quantitative analysis. Finally, we model "traditional technical progress" as capital-augmenting, whereas they consider it to be labor-augmenting and reversing automation.

A few theoretical papers have recently analyzed the distributional consequences of automation. Korinek and Stiglitz (2019) analyze the consequences of automation for income inequality, taking a broad view of the question. Similarly, Berg et al. (2018) study the implications of automation for growth and inequality in several related models. Our work differs from these both in some differences in our model, but mainly in our political economy and quantitative results. Guerreiro et al. (2022) examine optimal taxation of labor-substituting capital in a quantitative model with consideration of distributional implications, though they do not analyze this in a political economy framework, and their model differs from ours in many respects.

Our analysis of the political economy equilibrium under majority voting is related to a strand of the literature on optimal capital taxation. Judd (1985) and Chamley (1986) famously found that zero capital taxation is optimal in the long run. Lansing (1999) pointed out that this result does not hold for log utility, a result explained by Reinhorn (2019). More recently, Straub and Werning (2020) studied this problem and concluded that the original Chamley-Judd result only holds when entrepreneurs have an intertemporal elasticity of substitution below 1. We avoid these issues by focusing on the case of log utility, which is both highly tractable, and arguably more realistic, since it does not depend on manipulation of promised capital tax rates in the far future. Our analysis differs from earlier work in our application to automation. This results in novel results about the possibility of continuous growth through capital accumulation under complete automation, and a novel application to an episode of automation, which sheds light on the debate surrounding automation and UBI.

Organization The paper is organized as follows: Section 2 lays out the model. Section 3 analyzes the long-run outcomes of the model for fixed tax rates and analyzes comparative statics. Section 4 describes the political economy equilibrium of our model under majority voting. Section 5 presents a quantitative analysis of an episode of automation, contrasting the results under a fixed tax regime and the political economy equilibrium.

2 The Model

The model is set in continuous time with an infinite horizon and time denoted $t \geq 0$. There are four types of agents in the economy: workers who supply labor and consume their income every period; entrepreneurs, who own capital; firms, which rent capital, hire labor and produce goods; and the government, which levies various kinds of taxes, engages in wasteful spending, and arranges transfers to other agents.

2.1 Workers

There is a unit measure of workers with lifetime preferences

$$\int_0^\infty e^{-\gamma t} U\left(C_w(t), L(t)\right) dt, \tag{1}$$

where $C_w(t)$ is consumption and $L(t) \geq 0$ is labor supply. Workers supply $L(t)$ units of labor to firms, for which they receive an after-tax wage of $\left(1-\tau^\ell(t)\right) w(t)$. They also receive a transfer from the government, $T^w(t)$. They consume all of their income instantly, so their consumption is

$$C_w = \left(1-\tau^\ell\right) wL + T^w. \tag{2}$$

We assume that workers are not able to own capital. Therefore they exhibit hand-to-mouth consumption behavior by assumption. Their only non-trivial decision is labor supply, which satisfies the condition

$$\left(1-\tau^\ell\right) w U_C\left(C_w, L\right) \leq -U_L\left(C_w, L\right), \tag{3}$$

which holds with equality when $L > 0$.

A particularly useful utility function is log in consumption and leisure:

$$U(C,L) = \log(C) + \phi \log(1-L),$$

in which case labor supply satisfies

$$L = \frac{1}{1+\phi} \cdot \max\left(0, 1 - \frac{\phi T^w}{(1-\tau^\ell) w}\right).$$

The condition for zero labor supply is

$$\frac{T^w}{(1-\tau^\ell) w} \geq 1.$$

In other words, if the transfer is sufficiently large relative to the after-tax wage, the household does not supply any labor.

2.2 The Entrepreneurs

There is also a unit measure[5] of entrepreneurs who have preferences given by

$$\int_0^\infty e^{-\rho t} u(c_e). \tag{4}$$

Entrepreneurs each own capital k, which they rent to firms at rental rate r. Capital depreciates at rate δ, and capital income is subject to a tax of τ^k gross of depreciation. Therefore, an entrepreneur's budget constraint is given by

$$\dot{k} + c_e = r^k k + T^e, \tag{5}$$

where

$$r^k \equiv \left(1 - \tau^k\right) r - \delta \tag{6}$$

is the after-tax gross return on capital, and T^e is a transfer entrepreneurs receive from the government. Their consumption behavior satisfies the maximization condition

$$u'(c_e) = \lambda, \tag{7}$$

the co-state equation on capital

$$\dot{\lambda} = \lambda(\rho - r^k), \tag{8}$$

[5]The model could be modified so that the relative population of workers and entrepreneurs differ. However, a unit measure of each is the simplest.

and the transversality condition on capital

$$\lim_{t\to\infty} e^{-\rho t}\lambda k = 0. \tag{9}$$

2.3 The Firms

The firms choose an optimal production plan for given wage w and rental rate of capital r. The production technology is constant-returns-to-scale (CRS) and is given by

$$Y = \left[\int_0^1 \left(y\left(i\right)\right)^{1-\frac{1}{\sigma}} di\right]^{\frac{\sigma}{\sigma-1}}, \tag{10}$$

where $y\left(i\right)$ is the contribution of task i to overall output, Y. There is a unit measure of tasks indexed by $i \in [0,1]$. Each task can be performed by labor or capital. The output of task i satisfies:

$$y\left(i\right) = a\left(i\right) k\left(i\right) + b\left(i\right) \ell\left(i\right) \tag{11}$$

where $a\left(i\right)$ denotes the productivity of capital in task i, and $b\left(i\right)$ denotes the productivity of human labor in task i. We assume that $a\left(i\right), b\left(i\right) \geq 0$ for all i, and that tasks are ordered such that $a\left(i\right)/b\left(i\right)$ is weakly decreasing in i. We further require that $k\left(i\right), \ell\left(i\right) \geq 0$.

We adopt a task-based aggregation of the overall production process of the economy. This choice is more appropriate for our purposes as our focus is on automation and the process of automation is associated with substitution between capital and labor at the level of tasks, rather than goods. Therefore, tasks represent an intuitive disaggregated unit for the specification of the production technology.

2.4 An Aggregate Representation of the Production Process

The production technology specified in (10) depends on choices of capital and labor for each task i, as can be seen from (11). However, it is possible to simplify this representation using ideas from duality theory. The result is described in Proposition 1:

Proposition 1 (Aggregate Representation of Production)**.** *Suppose that the aggregate stocks of capital and labor are K and L respectively, and firms produce according to (10) and (11). Then cost-minimization by firms implies that aggregate production satisfies:*

$$Y \equiv F(K,L) = \left[\alpha^{\frac{1}{\sigma}} \left(A(\alpha)K\right)^{1-\frac{1}{\sigma}} + (1-\alpha)^{\frac{1}{\sigma}} \left(B(\alpha)L\right)^{1-\frac{1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, \tag{12}$$

where

$$A(\alpha) = \left[\frac{1}{\alpha}\int_0^{\alpha}\left(a\left(i\right)\right)^{\sigma-1} di\right]^{\frac{1}{\sigma-1}} \tag{13}$$

$$B(\alpha) = \left[\frac{1}{1-\alpha}\int_{\alpha}^{1}\left(b\left(i\right)\right)^{\sigma-1} di\right]^{\frac{1}{\sigma-1}} \tag{14}$$

and where $\alpha \in [0,1]$ *satisfies:*

$$\begin{cases} \frac{a(i)}{b(i)} \geq q(\alpha, K, L) & \text{for } i < \alpha \\ \frac{a(i)}{b(i)} \leq q(\alpha, K, L) & \text{for } i > \alpha. \end{cases} \tag{15}$$

where

$$q(\alpha, K, L) = \left(\frac{\int_0^{\alpha}\left(a\left(i\right)\right)^{\sigma-1} di}{\int_{\alpha}^{1}\left(b\left(i\right)\right)^{\sigma-1} di} \cdot \frac{L}{K}\right)^{\frac{1}{\sigma}}$$

Factor prices for labor and capital satisfy:

$$F_L(K, L) = w \tag{16}$$

$$F_K(K, L) = r. \tag{17}$$

Proof. See Appendix A.1 for details. □

Here α is the measure of tasks done by capital, i.e. tasks $i < \alpha$ are done by capital, and tasks $i > \alpha$ are done by labor, and $q = r/w$ is the relative factor cost of capital and labor. Since q is strictly increasing in α for given (K, L), whereas by assumption $a(i)/b(i)$ is weakly decreasing in i, (15) determines a unique value of α for given (K, L), and the aggregate production function $F(K, L)$ is well-defined.

Thus, the production function has a standard CES representation, with capital and labor productivity terms, for a given value of α. However, α itself varies endogenously with changes in technology and the supply of capital and labor. That is, we have a simple and parsimonious aggregate representation of the production process in which the task-level distribution of relative efficiency of capital and labor is succinctly captured by a single parameter identifying the marginal task that separates the tasks that are done by capital and labor respectively.[6]

[6] The expressions given in Proposition 1 are fairly standard, e.g. compare Proposition 1 in Acemoglu and Restrepo (2018c). However, we differ in the expression for $A(\alpha)$, as we allow capital productivity to vary across tasks, whereas they focus on the automation of tasks and labor-augmenting technological progress.

This has some important implications for changes in the productivity of capital. In the "standard case", in which all tasks may be done by only capital or labor, productivity changes only show up as changes in A (which in that case is independent of α) and, if $\sigma < 1$, this is labor-complementing technical progress. In our case, an increase in capital's productivity (an increase in the $a(i)$ schedule) also increases α and thus will necessarily have a labor-substituting effect. The relative distribution of these two (labor-complementing and labor-saving) effects, among other things, depends on the distribution of the technical progress across tasks. More broadly, this parameter/marginal task (α) quite intuitively captures the distinction between automation versus traditional technological progress. A change in the relative efficiency of capital and labor that shows up as a change in this parameter represents automation, and the orthogonal component represents traditional technical progress.

2.5 Government

The government sets taxes on capital income and labor income. It uses this income to pay for transfers to workers and entrepreneurs, and also to finance expenditures, G, which otherwise do not enter the model. The government budget constraint, therefore, is

$$\tau^{\ell} wL + \tau^{k} rK = T^{w} + T^{e} + G \tag{18}$$

In the political economy equilibrium considered in the paper, the government's tax rates are decided based on majority voting. Although, for simplicity, we assumed both entrepreneurs and workers have unit measure, we assume, on the margin and consistent with fact, workers are the majority and, hence, tax rates are chosen to maximize a worker's welfare.

2.6 Stationary Version of Model

For much of the paper, we will focus on a stationary version of the model, which will allow us to calculate a steady state and transition paths. This requires making a few additional assumptions. First, suppose that the capital and labor productivity of each task is time-invariant, *i.e.*,

$$a(i,t) = a(i) \quad \text{and} \quad b(i,t) = b(i) . \tag{19}$$

Further, suppose that government policies are also time-invariant, with a fixed share of GDP going to government spending, constant tax rates, and zero transfers to entrepreneurs:

$$G = \omega Y, \qquad \tau^{\ell}(t) = \tau^{\ell}, \qquad \tau^{k}(t) = \tau^{k}, \quad \text{and} \quad T^{e}(t) = 0. \tag{20}$$

where ω is the GDP share of government spending. Setting transfers to entrepreneurs to zero both simplifies the following discussion, and is consistent with the spirit of the subsequent politico-economic analysis where the focus is on the use of government policy for redistribution from entrepreneurs to workers.

Under these assumptions, the government budget constraint implies that transfers to workers, and consumption of workers and entrepreneurs respectively satisfy:

$$T^{w} = \tau^{\ell} w L + \tau^{k} r K - \omega Y, \tag{21}$$

$$C_{w} = w L + \tau^{k} r K - \omega Y, \tag{22}$$

$$c_{e} = r^{k} K - \dot{K}, \tag{23}$$

The other equations of the model are unchanged, except for time-invariance implied by (19) and (20).

3 Steady State and Comparative Statics

Before moving to the main part of our analysis, namely the quantitative analysis of an episode of automation in section 5.1 and the development of the political economy model in section 4, we present results on the existence of a steady state and comparative statics of the model. Since these results are fairly unsurprising and similar to others that have been previously obtained in the literature, we do not go into great detail, but instead highlight the key points and discuss their relation to the literature. Additional details of the derivations and discussion are given in the Appendix. Section 3.1 discusses conditions under which the economy experiences continuous growth rather than reaching a steady state. Section 3.2 describes the interior steady state, and presents comparative statics of automation and traditional technological progress for a particular stylized form of technology which we will use in our quantitative analysis. Finally, section 3.3 analyzes a stylized case in which automation may cause wages to decline, and gives a condition for this to occur. Throughout this section, we focus on the stationary version of the model described in section 2.6.

3.1 Steady state versus sustained growth

An interesting feature of models like ours[7] is the possibility of continuous growth through capital accumulation. In our model, the following condition determines whether this occurs:

Proposition 2. *Suppose that conditions (19) and (20) hold. Then the economy will achieve sustained growth if and only if:*

$$A(1) > \frac{\rho + \delta}{1 - \tau^k} \equiv r^* > 0 \tag{24}$$

where

$$A(1) = \left[\int_0^1 (a(i))^{\sigma-1} di\right]^{\frac{1}{\sigma-1}}$$

Proof. See Appendix A.2. □

Here $A(1)$ is capital productivity (13) when $\alpha = 1$, which equals the marginal product of capital as the share of tasks performed by capital goes to 1. Intuitively, $A(1)$ is the marginal product of capital as the capital-labor ratio goes to infinity. Since r is decreasing in K/L, it is also a lower bound on possible values of r. Meanwhile, the critical interest rate r^* is the steady state rental rate of capital, derived from the entrepreneur Euler equation. If $A(1) > r^*$, the rental rate is always above the steady state level, and so no steady state is reached. In this case, the production technology converges to an AK-model, with $Y = A(1) \cdot K$.[8]

Proposition 2 shows that the possibility of continuous growth depends on only three things: the value of $A(1)$, which is a function of technology, the discount rate of entrepreneurs, and the tax rate on capital. This implies that a shift in technology and/or tax policy may shift the economy from one growth regime to another. We elaborate on some of the consequences of this result in the following corollary:

Corollary 1. *Suppose that conditions (19) and (20) hold. Then, (i) If $a(i) > r^*$ for all i, then $A(1) > r^*$ and there is sustained growth. (ii) If $\sigma < 1$ and $a(i) = 0$ for a positive measure of tasks, then $A(1) = 0 < r^*$ and no long-run growth is possible. (iii) If $\sigma > 1$, a sufficient condition for sustained growth is that there exists m such that for all $i \in [0, m]$ we have $a(i) \geq m^{\frac{1}{1-\sigma}} r^*$.*

Proof. All results follow immediately from Proposition 2. □

[7] i.e. task-based CRS production with perfect substitutability between capital and labor at the task level.

[8] Analogous results have previously been obtained by other researchers using similar models. For example, Acemoglu and Restrepo (2018c) give a condition for continuous growth, though they do not have a term equivalent to our $A(1)$, as they do not allow capital productivity to vary over time or tasks, and do not include capital taxation. Aghion et al. (2017) consider the case that $\sigma < 1$ and $a(i) \in \{0, 1\}$, and discuss the possibility of continuous growth under full automation, i.e. when $\bar{\alpha} = 1$.

Result (i) gives a simple sufficient condition for sustained growth: sufficient capital productivity at all tasks. Result (ii) demonstrates the importance of the elasticity of substitution to whether continuous growth is possible. When $\sigma < 1$, each task is necessary, and thus sustained growth requires positive capital productivity at all tasks, excepting a measure 0 subset. In the absence of this, labor commands a positive share of income that is bounded away from zero and constrains the benefit of continued capital accumulation, and hence, sustained economic growth. This is related to Baumol's cost disease (Baumol and Bowen, 1966). Finally, result (iii) gives a sufficient condition for continuous growth under incomplete automation when $\sigma > 1$.

3.2 Steady States and Automation with Stylized Technology

We next examine how changes in the efficiency of capital/machines affects the distribution of income. For tractability and clarity of results, we focus on a stylized, but very useful, form of the $a(i)$ curve where

$$a(i) = \begin{cases} a & \text{for } 0 \leq i \leq \bar{\alpha} \\ 0 & \text{for } \bar{\alpha} < i \leq 1 \end{cases} \tag{25}$$

along with fixing $b(i) = 1$, $i \in [0, 1]$. Thus, capital can perform a fraction $\bar{\alpha}$ of tasks (for $i \in [0, \bar{\alpha}]$) at the same level of efficiency (relative to labor, $a(i)/b(i)$) denoted by efficiency parameter a. However, it cannot do other tasks, $i \in (\bar{\alpha}, 1]$ at all.[9]

In this stylized representation, traditional technical progress—which we construe as increased efficiency in performing tasks currently being perform by capital—can be viewed as an increase in parameter, a. In contrast, automation—which is considered to be the ability of capital to do tasks that earlier could only be done by labor—naturally can be thought of as an increase in $\bar{\alpha}$. Thus, in this task-based production representation, we can think of traditional technical progress as progress on the *intensive* margin (ability to better accomplish the tasks that can be already done) whereas automation can be thought of as progress on the *extensive* margin (ability to do tasks that could not be done earlier). Finally, to highlight the distinction between the nature of technical progress outlined above, without loss of generality, we assume that

$$A(1) < r^* < a \tag{26}$$

This ensures the existence of a steady state with positive capital.[10] We describe this steady

[9]This assumption about technology is similar to a common way of modeling automation in the literature, e.g. Acemoglu and Restrepo (2018c) and Aghion et al. (2017) model automation of tasks in an analogous way.

[10]These assumptions together imply $\bar{\alpha} < 1$, in which case $A(1) = 0$ if $\sigma \leq 1$, and $A(1) = \bar{\alpha}^{\frac{1}{\sigma-1}} a$ if $\sigma > 1$.

state in the following proposition:

Proposition 3. *Given that (25) holds, $b(i) = 1$, and that condition (26) holds, in steady state we have $\alpha = \bar{\alpha}$, and the capital-labor ratio, output, wage, and labor share satisfy:*

$$K/L = \bar{\alpha}\,(1-\bar{\alpha})^{\frac{1}{\sigma-1}} \left((r^*/a)^{\sigma-1} - \bar{\alpha}\right)^{\frac{\sigma}{1-\sigma}} /a > 0 \tag{27}$$

$$Y = \left((\bar{\alpha})^{\frac{1}{\sigma}} (aK)^{\frac{\sigma-1}{\sigma}} + (1-\bar{\alpha})^{\frac{1}{\sigma}} L^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \tag{28}$$

$$w = \left(\frac{1-\bar{\alpha}}{(r^*/a)^{\sigma-1} - \bar{\alpha}}\right)^{\frac{1}{\sigma-1}} \left(\frac{r^*}{a}\right) > 0 \tag{29}$$

$$s_L = 1 - \bar{\alpha}\,(r^*/a)^{1-\sigma} > 0 \tag{30}$$

where $s_L = wL/Y$ is the labor share.

Proof. See Appendix A.3. □

Note that while steady state capital and output are defined only up to steady state labor supply, which depends on equations (3) and (22) as well, the steady state wage and labor share do not depend on L. This is convenient, as we are chiefly interested in the distributional implications of automation. The expressions in Proposition 3 immediately yield the following comparative statics for traditional technical progress and automation for the wage and labor share:

Corollary 2. *A marginal increase in capital task productivity a increases the steady state wage, whereas the steady state labor share of income (i) increases for $\sigma < 1$; (ii) remains constant for $\sigma = 1$ (the Cobb-Douglas case); and (iii) decreases for $\sigma > 1$. In particular,*

$$\frac{dw}{da}\frac{a}{w} = \frac{\bar{\alpha}}{(r^*/a)^{\sigma-1} - \bar{\alpha}} > 0 \tag{31}$$

and

$$\frac{ds_L}{da} = \frac{\bar{\alpha}(1-\sigma)}{a}\left(\frac{r^*}{a}\right)^{1-\sigma} \gtreqqless 0 \qquad as \qquad \sigma \lesseqqgtr 1. \tag{32}$$

Proof. See Appendix A.3. □

Corollary 3. *A marginal increase in the automated share of tasks $\bar{\alpha}$ increases the steady state wage but decreases the steady state labor share. In particular,*

$$\frac{dw}{d\bar{\alpha}}\frac{\bar{\alpha}}{w} = \frac{1}{\sigma-1}\left(\frac{\bar{\alpha}}{1-\bar{\alpha}}\right)\left(\frac{1-(r^*/a)^{\sigma-1}}{(r^*/a)^{\sigma-1} - \bar{\alpha}}\right) > 0 \tag{33}$$

and

$$\frac{ds_L}{d\bar{\alpha}} = -\left(\frac{r^*}{a}\right)^{\sigma-1} < 0 \tag{34}$$

Proof. See Appendix A.3. □

Intuitively, traditional technical progress increases the wage, since this corresponds to increasing the effective capital supply, and capital is a complement of labor. It is likewise natural that changes in the labor share depend on σ, i.e. the degree of substitutability between capital and labor. As is well known, when $\sigma = 1$ the opposing substitution effect and the Edgeworth complementarity exactly cancel each other and the labor share is constant. When $\sigma > 1$, the substitution effect dominates causing the labor share to decline despite the increase in wages, whereas the reverse is true for $\sigma < 1$.

As shown in Corollary 3, automation (i.e. an increase in parameter $\bar{\alpha}$) also results in an increase in the steady state wage, mainly due to the increase in steady state capital. However, the effect of automation on the labor share, unlike that of traditional technical progress, does not depend on the elasticity of substitution σ. This is a vital difference that speaks to the current debate centered around automation and associated policy proposals like universal basic income. The key to this difference lies intuitively in the fact that the effect of automation manifests as substitution *within* tasks (or at the task level), whereas the effect of traditional technical progress (a) operates *between* tasks (or across tasks at the aggregate level). Thus, the distributional effect of automation are mainly driven by perfect substitutability of capital and labor within a task whereas the effect of traditional technical progress depend on the degree of substitutability of capital and labor between tasks (*i.e.*, σ). Moreover, since this difference between the degree of substitutability (between capital and labor) within and between tasks is larger when $\sigma < 1$, the qualitative effects of automation compared to traditional technical progress are worse in this case compared to when $\sigma > 1$. This is substantiated by the fact that when $\sigma < 1$ an increase in a increases the labor share of income whereas it falls for an increase in $\bar{\alpha}$, but when $\sigma > 1$ labor share of income falls in both scenarios.[11]

3.3 Variable Labor Productivity and Wage Declines

The result in Corollary 3 that automation necessarily raises wages in the long run may be taken as evidence against concerns about automation harming human labor. However,

[11] Acemoglu and Restrepo (2018c) obtain similar results in a model of automation with balanced growth. They find that a permanent shift towards a higher share of automated tasks results in a decline in the labor share, whereas the wage may be higher or lower in the short run, but is generally higher or approximately the same in the long run.

this result depended on our assumption of constant labor productivity across tasks. By contrast, if labor productivity differs across tasks, and relatively high productivity tasks are automated, automation may lower wages.

To illustrate this possibility, suppose that labor productivity $b(i)$ is:

$$b(i) = \begin{cases} b_m & i \in [0, \bar{\alpha}_1] \\ b_1 & i \in (\bar{\alpha}_1, 1] \end{cases} \tag{35}$$

We continue to assume that capital productivity is piecewise across tasks and given by (25). Then we consider an episode of automation that replaces the marginal tasks with productivity b_m. The effect on the steady state wage is given in the following proposition:

Proposition 4. *Suppose that capital productivity is given by (25) and labor productivity by (35). Suppose that initially the economy is in steady state with $\bar{\alpha} = \bar{\alpha}_0 < \bar{\alpha}_1$, and then there is a one-time shift in the capital productivity curve to $\bar{\alpha} = \bar{\alpha}_1$. Then the steady state wage declines if and only if:*

$$\frac{b_m}{b_1} > \left[\frac{(a/r^*)^{1-\sigma} - \bar{\alpha}_1}{1 - \bar{\alpha}_1}\right]^{\frac{1}{1-\sigma}} \tag{36}$$

Proof. See Appendix A.4. □

As we previously assumed $a > r^*$, condition (36) will never be satisfied for $b_m \leq b_1$, i.e. when the productivity of tasks automated is equal or lower than the productivity of remaining tasks. However, when $b_m > b_1$, this condition may be satisfied, and thus it is possible for the wage to decline in the long run. Whether this condition holds depends on the relative productivity of the automated tasks to the remaining tasks b_m/b_1, the degree of substitutability between capital and labor σ, and the ratio of capital task productivity to the steady state rental rate of capital a/r^*. The last matters because this determines the level of steady state capital.

4 Government Policy under Majority Voting

To examine the interplay of policy and technical change alluded to earlier, we endogenize government policy by considering a political economy model with majority voting, where worker households are in the majority.

4.1 Statement and Solution of the Planning Problem

Consider the model outlined previously in Section 2. Recall that we set government spending equal to a fixed share ω of output Y, and set transfers to entrepreneurs to zero. This is consistent with the spirit of our politico-economic analysis, which focuses on the use of government policy for redistribution from entrepreneurs to workers. We simplify the analysis by focusing on cases in which entrepreneurs have log utility:

$$u(c_e) = \log(c_e)$$

The policy is set by majority voting with full commitment. Since we assume that workers are in the majority, this amounts to setting policy to maximize the welfare of workers. Thus we can formulate the problem as a Ramsey Planning problem. There is only one state variable in the model, the capital stock, K. Thus, the problem of the Ramsey Planner is to set a path of $\{C, L, K\}$ subject to the resource constraint and implementability conditions, which we derive next.

We would like to reduce the model to a minimal set of restrictions on the choices of main variables $\{C_w, c_e, L, K\}$. Output, r, and w are defined for given (K, L) as

$$F(K, L), \quad r(K, L), \quad w(K, L).$$

Next observe that choosing τ^k will imply a choice of r^k according to (6). We can therefore treat $\left\{r^k\right\}$ as a choice variable of the planner. Once the path of $\left\{r^k\right\}$ is given, we are left with the two conditions, (8) and (23), which, together with the definition of λ, can be combined into a single implementability condition. However, given our assumption of log utility, we can simplify the problem further, since in this case we can solve for the decision rule of the entrepreneurs exactly. Under log utility, entrepreneur consumption satisfies:

$$c_e(t) = \rho K(t) \tag{37}$$

Next, we observe that the planner can (essentially) determine the level of labor supply by altering the tax on labor income, so as to satisfy workers' labor supply condition, (5). Finally, we observe that the remaining conditions only enter as part of the resource constraint. In particular, combining (22), (21), and (23), we get

$$C_w + c_e + \dot{K} = (1-\omega)F(K, L) - \delta K. \tag{38}$$

We can therefore state the Ramsey Planning Problem for the social planner which maximizes a worker's life-time utility under majority voting as

$$\max_{\{C_w,c_e,K,L\}} \left\{ \int e^{-\gamma t} U\left(C_w, L\right) \right\}, \tag{39}$$

subject to (37), (38),and

$$L \geq 0 \tag{40}$$

Relation to Literature on Optimal Capital Taxation. This problem is closely related to a well-studied problem in the theory of optimal capital taxation. The formulation of the problem here is nearly identical to that in Judd (1985), more recently studied by Straub and Werning (2020), with three chief differences.[12] First, we admit the possibility of relatively impatient workers ($\gamma > \rho$); second, we allow the possibility of continuous growth through automation; and third, we restrict attention to the case of log utility. The last assumption makes the problem analogous to that analyzed by Lansing (1999), and avoids the issue of the non-existence of interior steady states discussed by Straub and Werning (2020).

4.2 Solving the Ramsey Planning Problem

We can simplify the statement of the Ramsey problem by substituting the entrepreneur's behavioral rule (37) directly into the resource constraint (38). The Hamiltonian for the Ramsey Planning problem is then:

$$\mathcal{H} = U\left(C_w, L\right) + \lambda\left[(1-\omega)F\left(K, L\right) - \delta K - \rho K - C_w\right] + \mu_L L$$

The choice variables are (C_w, L) and the state variable is K. This yields optimality conditions:

$$U_C\left(C_w, L\right) = \lambda \tag{41}$$

$$-U_L\left(C_w, L\right) \geq \lambda(1-\omega)F_L\left(K, L\right) \tag{42}$$

$$-\dot{\lambda}/\lambda = (1-\omega)F_K\left(K, L\right) - \delta - \rho - \gamma \tag{43}$$

In addition, the entrepreneur consumption rule (37) and aggregate resource constraint (38) must hold. These five equations, along with with complementary slackness conditions on L, determine the four choice variables of the Ramsey problem and the Lagrange multiplier λ.

[12]We also formulate the problem in continuous time, whereas they use discrete time, but this should not qualitatively alter any result.

Labor Taxation. We begin with the implications for labor taxation. Combining equations (41) and (42) yields:

$$-U_L(C_w, L) \geq (1-\omega)U_C(C_w, L) \cdot F_L(K, L) \tag{44}$$

which holds with equality when $L > 0$. Comparing with (3) and (78), this implies that the labor tax equals the government expenditure share, $\tau^\ell = \omega$. Note that this implies a zero tax on labor when $\omega = 0$. Intuitively, in the absence of government spending linked directly to GDP, workers would not choose to distort the labor market at all, since at the margin the only purpose of taxing would be to transfer funds to themselves. By contrast, when government spending is a share of GDP, a fraction of any increase in production goes to increased government spending, which has no benefit to workers and thus they perceive as a tax on their efforts. Thus they optimally choose to internalize this cost by setting the labor tax equal to the share of GDP consumed by the government.

Capital Taxation. We can likewise derive an expression for the capital tax rate. Substituting the entrepreneur consumption rule (37) into the entrepreneur Euler equation (8), and combining with the optimality condition (43), we obtain:

$$1 - \tau^k = (1-\omega)\left(\frac{\dot{K}/K + \delta + \rho}{\rho + \gamma + \delta - \dot{\lambda}/\lambda}\right) \tag{45}$$

We immediately observe several things about the optimal capital tax rate. First, it is higher when the government expenditure share ω is higher. This is unsurprising, given our discussion of the same result for labor taxes. Second, it is increasing in the workers' discount rate γ. Intuitively, the benefits of capital taxes are obtained immediately, in the form of higher consumption, while the costs are lower future consumption due to a reduced future capital stock. The more impatient are workers, the more these future costs are discounted, and thus the higher will be optimal capital taxes. Third, the optimal tax rate is decreasing in the growth rate of the capital stock. This is because a growing capital stock indicates that the economy is below its steady state efficient level of capital, indicating higher marginal product of capital and a greater return to investment. Finally, the optimal tax rate is increasing in the growth rate of worker consumption (i.e. decreasing in the growth rate of λ). This is because workers would like to smooth their consumption, and thus respond to an increasing path of consumption by raising capital taxes in order to boost their current consumption.

4.3 Steady state and balanced growth

We next examine what happens in the long run. Specifically, we ask under what conditions the economy achieves a steady state vs. continuous growth, and if the latter whether it achieves a balanced growth path. We also characterize the resulting steady state or growth path.

Steady state. First suppose that a steady state is reached. In a steady state, equation (43) implies:

$$F_K(K, L) = \frac{\delta + \rho + \gamma}{1 - \omega} = r^*$$

That is, the marginal product of capital reaches a specific level, r^*. Note that r^* is increasing in $\delta + \rho + \gamma$ and in ω, implying that steady state capital is decreasing in all of these, as one would expect.

From (45), the steady state capital tax rate is:

$$\tau_{ss}^k = 1 - (1 - \omega)\left(\frac{\delta + \rho}{\rho + \gamma + \delta}\right) = \omega + \frac{\gamma}{r^*} \tag{46}$$

Interestingly, the steady state capital tax rate is independent of technology, and only depends on the government consumption share, depreciation, and the discount rates of entrepreneurs and workers. Note further that the steady state capital tax rate is always higher than ω, and therefore higher than the labor tax rate.[13]

When is the steady state reached? From Proposition 2, we know that a steady state will be reached iff $A(1) < r^*$. We next turn to the question of what happens when this condition is not satisfied.

Continuous Growth with CRRA utility. Next we consider the possibility of continuous growth through the accumulation of capital. Here we make a further simplifying assumption: suppose that worker utility is separable in consumption and labor, and is CRRA in consumption, i.e.:

$$U(C_w, L) = \frac{C_w^{1-\psi}}{1 - \psi} - h(L)$$

Now suppose we reach a point after which $L = 0$ and the economy enjoys continuous

[13] This differs from the famous Chamley-Judd result of zero steady state capital taxation, derived separately by Chamley (1986) and Judd (1985). That this result does not hold under log utility was first pointed out by Lansing (1999). Straub and Werning (2020) have since clarified that the Chamley-Judd result only occurs when entrepreneurs have IES > 1, and no finite steady state exists when IES < 1.

growth. When $L = 0$, the production function is AK with:

$$Y = A_1 K \tag{47}$$

where $A_1 = A(1)$. The marginal product of capital is likewise

$$F_K(K, 0) = A_1 \tag{48}$$

Thus when $L = 0$, the growth of the capital stock and worker consumption satisfy:

$$\frac{\dot{K}}{K} = (1-\omega)A_1 - \delta - \rho - \frac{C_w}{K} \tag{49}$$

$$\frac{\dot{C}_w}{C_w} = \frac{(1-\omega)A_1 - \delta - \rho - \gamma}{\varphi} \tag{50}$$

We now ask whether a balanced growth path exists. A balanced growth path requires that all the variables $\{C_w, c_e, K\}$ grow at a constant rate g. If this is true for K, it will also be true for c_e, since $c_e = \rho K$. Thus we only need to check that worker consumption C_w and capital K grow at the same rate g. We can use equation (50) to solve for g, which satisfies:

$$g = \frac{(1-\omega)A_1 - \delta - \rho - \gamma}{\varphi} \tag{51}$$

A balanced growth path requires a positive growth rate. Thus (51) implies that a necessary condition for a balanced growth path is:

$$A_1 > \frac{\delta + \rho + \gamma}{1-\omega} = r^* \tag{52}$$

which is just the same condition as found in Proposition 2 for continuous growth.

Now we must also check that this growth rate is consistent with the economy's resource constraint (49). By setting $\dot{K}/K = g$ in this equation, we can solve for worker consumption C_w as a multiple of capital K. Doing so, we find that worker consumption will be:

$$\frac{C_w}{K} = [(1-\omega)A_1 - \delta - \rho]\left(\frac{\varphi - 1}{\varphi}\right) + \frac{\gamma}{\varphi} \tag{53}$$

Note that for φ sufficiently low, (53) implies $C_w < 0$. What is going on in this case? When φ is low, workers have little desire to smooth consumption, and thus choose to grow the capital stock quickly to increase the growth rate. For sufficiently low φ, the growth rate becomes fast enough that, on the balanced growth path, the integral of worker lifetime utility diverges. Thus the planning problem (39) is not well-defined.

To see this, observe that the condition $C_w > 0$, where C_w is defined by equation (53), can be written as:

$$(1-\varphi)g - \gamma < 0$$

If C_w is growing at rate g, then discounted utility $(1-\varphi)^{-1}e^{-\gamma t}C_w^{1-\varphi}$ is growing at rate $(1-\varphi)g - \gamma$, which must be a negative number for the integral to converge. Thus for the planners problem to be well-defined, we need the parameters of the problem to satisfy:

$$(1-\varphi)\left[(1-\omega)A_1 - \delta - \rho\right] < \gamma \tag{54}$$

which we will assume holds.

The capital tax rate along a balanced growth path will be:

$$\tau_{bg}^k = \omega + \frac{(\varphi - 1)\, g + \gamma}{A_1} \tag{55}$$

Note that the tax rate will always be greater than ω as long as condition (54) is satisfied. Further, this condition is the same as (46) with $g = 0$ and $A_1 = r^*$.

We can compare our solution to the optimal growth problem with a representative agent (i.e. that owns capital and supplies labor). In that case, if a balanced growth path is achieved, it would feature growth rate:

$$g = \frac{(1-\omega)A_1 - \delta - \rho}{\varphi}$$

which corresponds to capital taxation equal to $\tau^k = \omega$. Thus growth will be slower, and capital taxes higher, in this case than with a representative agent.

We can summarize our results in the following proposition:

Proposition 5. *Suppose that entrepreneurs have log utility in consumption. Then:*

(i) If $(1-\omega)A(1) < \rho + \gamma + \delta$, then the economy will reach a steady state with positive labor supply $L > 0$, at which capital satisfies:

$$F_K(K, L) = \frac{\rho + \delta + \gamma}{1 - \omega} \equiv r^*$$

and the capital tax rate satisfies:

$$\tau_{ss}^k = 1 - (1-\omega)\left(\frac{\delta + \rho}{\rho + \gamma + \delta}\right) = \omega + \frac{\gamma}{r^*}$$

(ii) If $(1-\omega)A(1) > \rho+\gamma+\delta$ *and workers have utility that is separable in* L *and CRRA in consumption with CRRA parameter* ψ*, which satisfies:*

$$\varphi > 1 - \frac{\gamma}{(1-\omega)A(1)-\delta-\rho}$$

Then a balanced growth path exists, with growth rate:

$$g = \frac{(1-\omega)A(1)-\delta-\rho-\gamma}{\varphi}$$

and capital tax rate:

$$\tau_{bg}^{k} = \omega + \frac{(\varphi-1)\,g+\gamma}{A(1)}$$

4.4 Extension: Robot Taxes

The only policies considered so far have been taxes on capital income and on labor income overall. Given that our concern is with automation in particular, a natural question is whether the workers may wish to tax some kinds of capital in different rates than others. For example, they may wish to tax those kinds that are competing most directly with them, what one might call a "robot tax".

We can address this question in our setting by allowing capital taxes to differ with respect to different tasks. For simplicity, suppose we decompose tasks into two types, 1 and 2; that is, we make subsets $U_1, U_2 \subset [0,1]$ such that $U_1 \cup U_2 = [0,1]$, $U_1 \cap U_2 = \varnothing$. We further assume that both sets are integrable, so that e.g. $\int_{U_1} k(i)$ is well defined. We allow the rental rate of capital to differ between the two types. Then we can define aggregate production as a function of aggregate labor L and aggregate capital assigned to the two types (K_1, K_2) as:

$$\begin{aligned} F\left(K_1, K_2, L\right) = & \max_{\{k(i),\ell(i)\}} \left\{ \left[\int_0^1 \left(a\left(i\right)k\left(i\right) + b\left(i\right)\ell\left(i\right)\right)^{1-\frac{1}{\sigma}} di \right]^{\frac{\sigma}{\sigma-1}} \right\} \qquad (56) \\ \text{s.t.} \quad & \int_i \ell\left(i\right) \leq L, \ \int_{U_1} k\left(i\right) \leq K_1, \ \int_{U_2} k\left(i\right) \leq K_2 \\ & k\left(i\right) \geq 0, \ \ell\left(i\right) \geq 0 \end{aligned}$$

We don't need to work out the aggregate representation of production in more detail to solve the optimal policy problem, since we only need the aggregate production function to do so. Given the aggregate production function, firms solve:

$$\max_{K_1,K_2,L} \{F(K_1,K_2,L) - r_1K_1 - r_2K_2 - wL\} \tag{57}$$
$$\text{s.t.} \quad K_1 \geq 0,\ K_2 \geq 0,\ L \geq 0$$

which yields optimality conditions with respect to capital of:

$$F_{K_1}(K_1,K_2,L) + \phi_1 = r_1 \tag{58}$$

$$F_{K_2}(K_1,K_2,L) + \phi_2 = r_2 \tag{59}$$

where ϕ_1, ϕ_2 are the Lagrange multipliers on the non-negativity constraints $K_1 \geq 0$ and $K_2 \geq 0$ respectively.

Now there is an additional decision relative to our baseline problem. In our baseline problem, entrepreneurs simply chose their overall stock of capital K, which earned a given return r^k. Now, however, entrepreneurs can allocate their stock of capital K between two uses K_1, K_2, which may yield different returns. Since capital is perfectly mobile across tasks, they will choose K_1, K_2 to maximize their total return:

$$R^k(K) = \max\left\{\left(1-\tau_1^k\right) r_1K_1 + \left(1-\tau_2^k\right) r_2K_2\right\} \tag{60}$$
$$\text{s.t.} \quad K_1 + K_2 \leq K,\ K_1 \geq 0,\ K_2 \geq 0$$

This problem yields first-order condition:

$$\left(1-\tau_1^k\right) r_1 + \phi_1' = \left(1-\tau_2^k\right) r_2 + \phi_2' \tag{61}$$

where ϕ_1', ϕ_2' are again the Lagrange multipliers on the non-negativity constraints $K_1 \geq 0$ and $K_2 \geq 0$ respectively.

Substituting (58) and (59) into (61) yields:

$$\left(1-\tau_1^k\right) F_{K_1}(K_1,K_2,L) + \tilde{\phi}_1 = \left(1-\tau_2^k\right) F_{K_2}(K_1,K_2,L) + \tilde{\phi}_2 \tag{62}$$

where:

$$\tilde{\phi}_1 = \phi_1' + \left(1-\tau_1^k\right)\phi_1$$
$$\tilde{\phi}_2 = \phi_2' + \left(1-\tau_1^k\right)\phi_2$$

Now observe that, by varying taxes on the two kinds of capital, the planner can effectively shift the allocation of capital between K_1 and K_2. As before, the planner can vary labor taxes to effectively choose L, subject to non-negativity constraints.

The planner's problem is therefore to choose the paths of (C_w, c_e, K_1, K_2, L) to maximize worker utility, subject to the problem's constraints. As before, we can substitute $c_e = \rho K$ directly into the resource constraint, and we can also substitute $K = K_1 + K_2$ directly in. This yields resource constraint:

$$C + \dot{K} \leq F\left(K - K_2, K_2, L\right) - \delta K - \rho K \tag{63}$$

This, together with non-negativity constraints on L, and the constraint that $K_2 \in [0, K]$ (which gives us the non-negativity constraints on K_1 and K_2), are the full set of constraints for the planner's problem.

Given this problem, we can easily prove the following proposition:

Proposition 6. *The planner never chooses to set different tax rates for different types of capital. Therefore $\tau_1^k = \tau_2^k$.*

Proof. The Hamiltonian of the problem is:

$$H = U\left(C, L\right) + \lambda\left\{F\left(K - K_2, K_2, L\right) - \delta K - \rho K - C\right\} + \mu L + \phi_1\left(K - K_2\right) + \phi_2 K_2$$

The optimality condition with respect to K_2 is:

$$\lambda F_{K_1} + \phi_1 = \lambda F_{K_2} + \phi_2$$

where ϕ_1 and ϕ_2 are the non-negativity constraints on K_1 and K_2 respectively. Comparing this equation to (62), we can immediately see that this condition is satisfied when there is an equal tax rate across the two kinds of capital, i.e. $\tau_1^k = \tau_2^k$. □

Thus we see that workers will never prefer to set different tax rates between the two kinds of capital. Moreover, since the partition of tasks between K_1 and K_2 was arbitrary, this extends to any distribution of tax rates across different kinds of capital. Workers will always choose to tax different kinds of capital at the same rate. Thus there will be no robot taxes.[14]

[14] Guerreiro et al. (2022) consider whether robots should be taxed in a model with automation and endogenous skill choice. They also find that optimal robot taxes are zero in the long run. However, unlike us, they find that non-zero robot taxes are optimal in the short run, since workers are specialized in different tasks. Our setup differs from theirs since we assume perfect labor mobility across tasks.

5 Quantitative Analysis of an Episode of Automation

We now turn to a quantitative analysis of an episode of automation. We begin by analyzing such an episode under a regime of fixed tax rates, as analyzed in section 3. This allows us to set a baseline for how how wages, the labor share, and welfare changes in response to an episode of automation, both between steady states and over the transition. We then analyze the same episode with policy set according to majority voting, as described in section 4.

5.1 Episode of Automation with Fixed Taxes

We begin by assuming that labor and capital tax rates are fixed. We consider a one-time automation represented by a one-time increase in $\bar{\alpha}$ from $\bar{\alpha}_o$ to $\bar{\alpha}_1 > \bar{\alpha}_o$ for some fixed $a > r^*$ starting in a steady-state. Moreover, to have a more realistic assessment of the dynamics, the increase in $\bar{\alpha}$ is allowed to occur gradually. In particular, $\bar{\alpha}$ follows the following exogenous process:

$$\dot{\bar{\alpha}} = \theta(\bar{\alpha}_1 - \bar{\alpha}) \tag{64}$$

with $\bar{\alpha} = \bar{\alpha}_0$ initially.

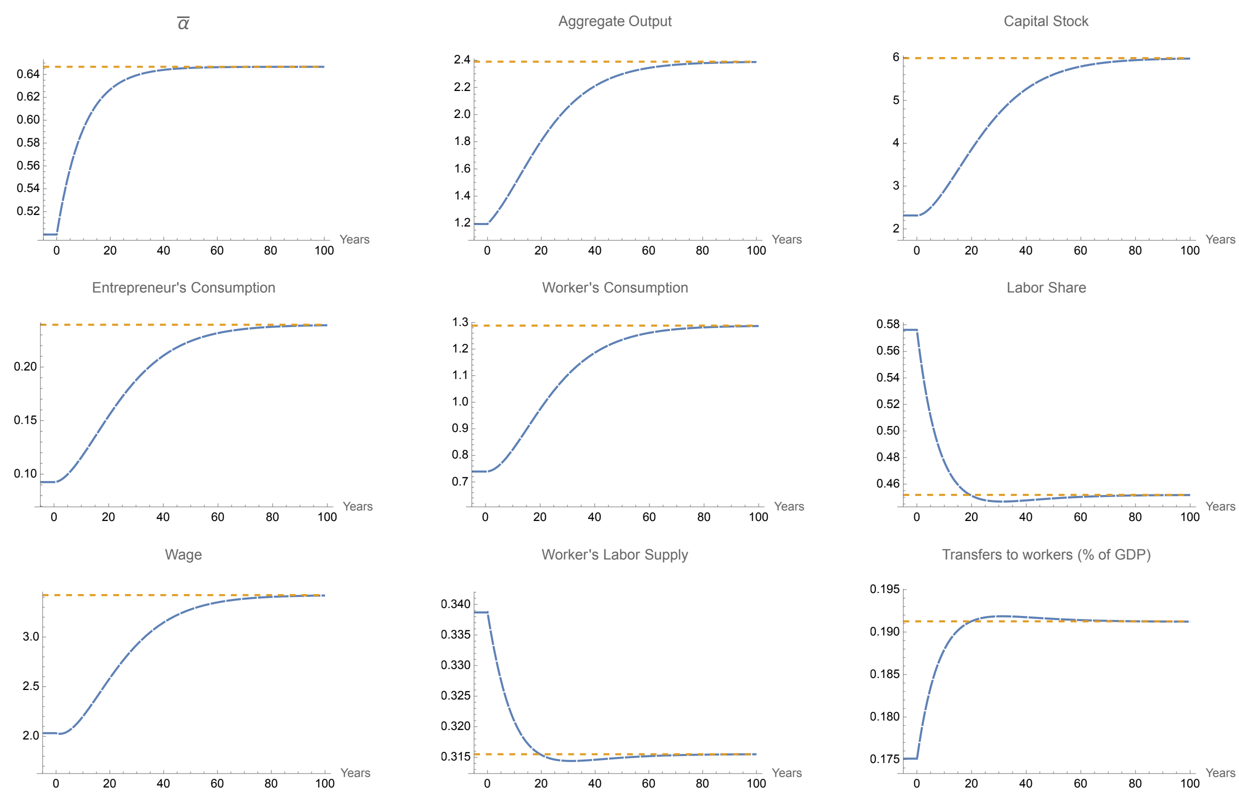
The results are presented for values of σ below and above 1. In each case, we choose parameter values that are very standard in the macro literature and/or based on data. We calibrate the model so that the initial steady state is typical in terms of tax rates, ratio of consumption to output, labor income share, and ratio of capital to output, among other variables. For both values of σ we set $\rho = .04$ which gives an annual real interest rate of 4% and set γ to be .06. $\tau^k = .36$ and $\tau^\ell = .23$ are based on Fernández-Villaverde et al. (2015). For ϕ we choose a value of 1.4, which implies $L = .339$. Similarly, $\delta = .1$ corresponds to a 10% depreciation rate for capital. These values are standard in the literature and, recall, $b(i) = 1$.

In order to calibrate the model, a number of parameters need to be changed with σ. For $\sigma = .8$, we set $\bar{\alpha}_0 = .5$, but when $\sigma = 1.2$, $\bar{\alpha}_0$ needs to be set to .25. We calibrate $\bar{\alpha}_1$ to achieve an identical 100% increase in output (*i.e.,* doubling of output) for both cases, which requires an increase in $\bar{\alpha}$ from .5 to .6467 and from .25 to .3107 respectively. Furthermore, the parameter a for labor productivity is respectively set to .5 and 3.

The calibrated steady state is very similar in both cases with a labor share of income of about 57.7%. Consumption is about 78% of private sector income, and the capital-to-output ratio is about 1.93.

Finally, to have similar transitional dynamics for both cases of σ, θ is set to an identical value of .1 which implies that the impact of automation lasts for a couple of decades—a typical duration for a general purpose technology. During the transition, tax rates are kept constant and transfers to workers adjust to respect the government's budget constraint. Recall, transfers to entrepreneurs have been set to zero.

Figure 1: Dynamics for Gradual Increase with $\sigma = .8$

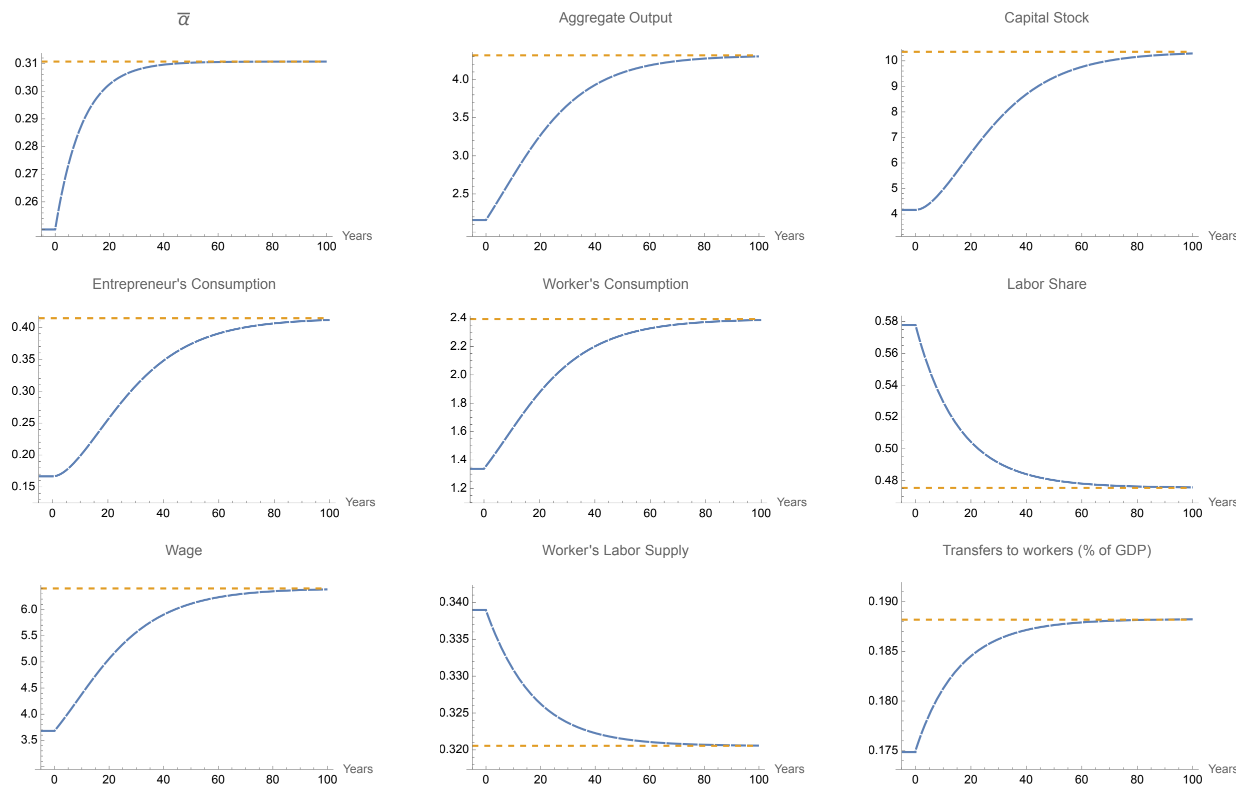


The results for the transitional dynamics are shown in Figure 1 for $\sigma = .8$ and in Figure 2 for $\sigma = 1.2$.[15] Before turning to the dynamics, it is instructive to look at changes across steady states in the two cases.

The economy is initially in the steady state corresponding to constant tax rates of $\tau^k = .36$ and $\tau^\ell = .23$. Then an episode of automation occurs, in which $\bar{\alpha}$ increases over a period of time. This causes output to approximately double, as calibrated, driven by a large increase in capital. The labor share of income falls in both cases as shown earlier. For $\sigma = .8$, it goes down from 57.6% to 45.2% whereas the decline is a bit less from 57.8% to 47.5% for $\sigma = 1.2$; and it is accompanied by respective increases in wages of 68.4% and

[15] As the changes across steady states are large (because output doubles) accuracy of numerical solutions requires solving the model by nonlinear solution techniques. For this purpose, we use the reverse-shooting algorithm of Atolia and Buffie (2009).

Figure 2: Dynamics for Gradual Increase with $\sigma = 1.2$



74.0%. This outcome must be differentiated from the observation made at the end of Section 3.2 (following Corollary 3) that the differential effect of automation (relative to traditional technical progress) on the labor share of income is worse when $\sigma > 1$. Here, it is shown that the effect of automation itself on the labor share of income is slightly better when $\sigma > 1$.

The slight difference in labor share of income for the two values of σ also show up in the consumption of workers, which increases respectively by 74.3% and 78.9% for σ equal to .8 and 1.2, less than the 100% increase in output. Thus workers are slightly better off in terms of consumption for the higher value of σ. However, the relatively larger gain in consumption for a higher σ is offset by a relatively smaller reduction in disutility from labor, because labor supplied falls by 5.4% compared to a 6.8% fall when $\sigma = .8$.

The transition itself is generally smooth and monotonic for all the macro variables, as seen in Figures 1 and 2, except for some variables (labor share, labor supply, and transfers to workers) for $\sigma = .8$. As is usual in neoclassical models, it takes more time for the macro variables to transition compared to the exogenous change in $\bar{\alpha}$ due to the endogenous dynamics of capital accumulation. Overall, the time taken for endogenous dynamics to play out is about twice that of the exogenous dynamics of automation. For workers, welfare increases by a consumption equivalent of 24.6% for lower the value of σ, and 29.4% for the higher value. For entrepreneurs, these numbers are 57.7% and 49.0% respectively.[16]

5.2 Quantitative Effects of Automation under Majority Voting

We now turn to quantifying the effects of automation under majority voting for the $a(i)$ curve in (34) with the purpose of comparing and contrasting the outcomes in Section 5.1 under fixed taxes. As mentioned earlier, the specification allows an easy and intuitive separation of the effects of traditional technical progress and automation. Once again, we consider a gradual increase in $\bar{\alpha}$. The specification of the model and calibration remain the same as in Section 5.1, and so those details are omitted. To facilitate comparison with the results in Section 5.1, we assume that we start in the steady state with fixed taxes, and there is an identical change in $\bar{\alpha}$ as well. Thus the only departure from the case analyzed in Section 5.1 is that taxes are set according to majority voting starting at $t = 0$.

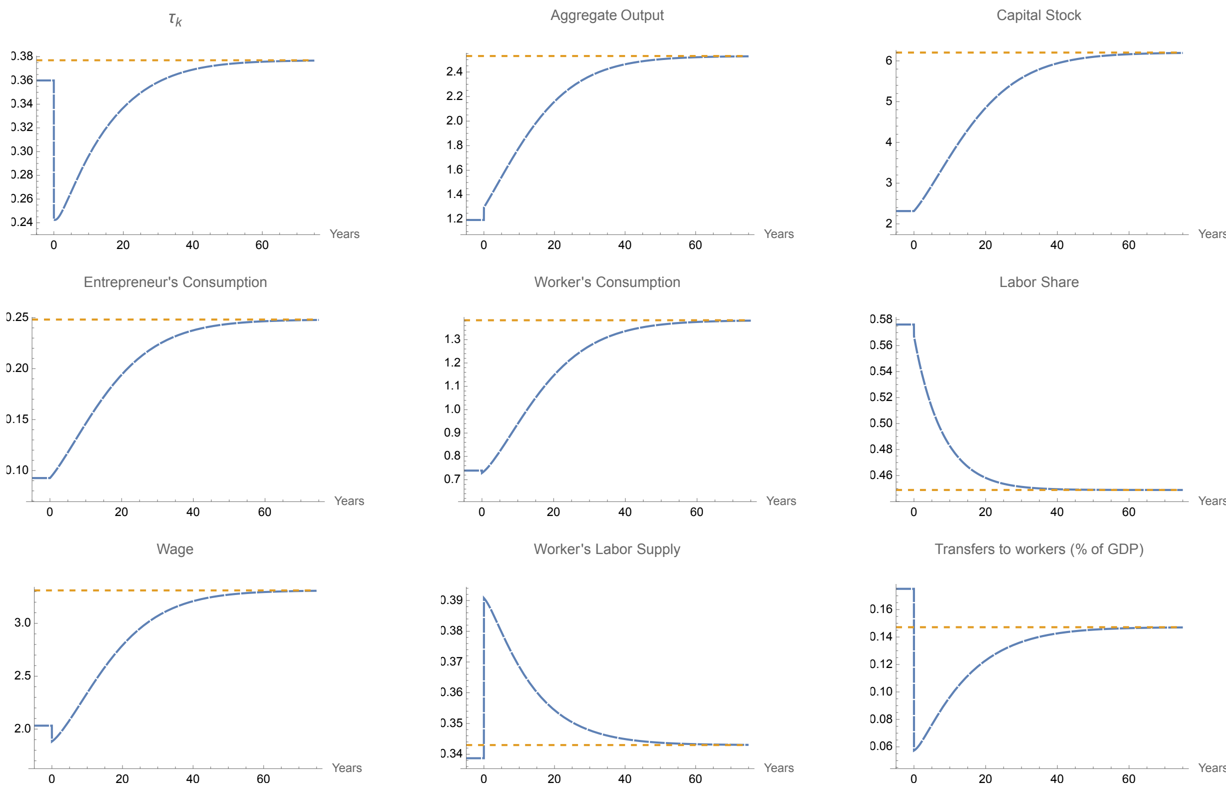
Table 1: Steady States for Competitive and Majority Voting Cases

	$\sigma = .8$			$\sigma = 1.2$		
	Initial	Fixed Tax	Majority	Initial	Fixed Tax	Majority
Y	1.195	2.389	2.530	2.159	4.318	4.548
C_w	0.739	1.288	1.383	1.338	2.393	2.570
c_e	0.093	0.239	0.248	0.167	0.414	0.422
w	2.302	3.422	3.311	3.680	6.404	6.218
L	0.339	0.315	0.343	0.339	0.321	0.350
K	2.314	5.987	6.205	4.165	10.535	10.560
labor share	0.576	0.452	0.449	0.578	0.475	0.478
capital share	0.424	0.548	0.551	0.422	0.525	0.522
K/Y	1.937	2.506	2.452	1.929	2.398	2.322
C_w/Y	0.619	0.539	0.547	0.620	0.554	0.565
c_e/Y	0.077	0.100	0.098	0.077	0.096	0.093
G/Y	0.110	0.110	0.110	0.110	0.110	0.110
$\bar{\alpha}$	0.500	0.6471	0.6471	0.250	0.3107	0.3107
τ^k	0.360	0.360	0.377	0.360	0.360	0.377
τ^ℓ	0.230	0.230	0. 110	0.230	0.23	0.110
T^w/Y	0.175	0.191	0.147	0.175	0.188	0.139
T^w/Y adjusted	0.106	0.137	0.147	0.106	0.131	0. 139

Table 1 summarizes the important macro variables, ratios, and numbers for various steady states for both values of σ. In particular, for each value of σ there is information about

[16]These numbers are the percentage increases in consumption in the initial steady state (with labor supply fixed at its initial level) that would provide workers and entrepreneurs with equivalent welfare as in the episode of automation.

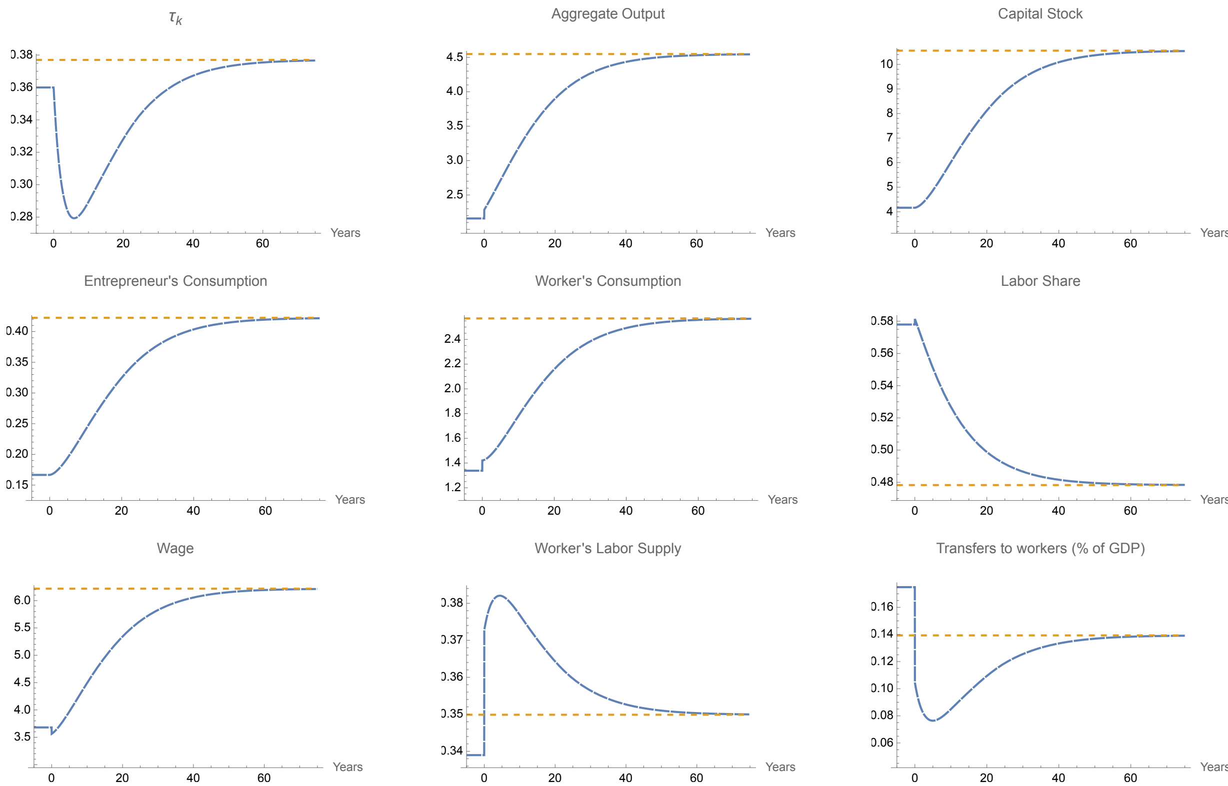
Figure 3: Dynamics for Gradual Increase with $\sigma = .8$ under majority voting



three steady states: the initial steady state, the final steady state with automation under competitive equilibrium (discussed in Section 5.1) and the final steady state under majority voting.

As the changes in going from the fixed tax regime to majority voting are very similar, both qualitatively and quantitatively, for both σ equal to .8 and 1.2, we restrict our discussion to the low σ case. Comparing the fixed tax and majority voting outcomes, note that under majority voting, steady state $\tau^k = 0.377$, i.e., slightly higher than our calibration based on current tax rates, whereas $\tau^\ell = 0.11$, i.e., much lower. This suggests that the original tax system was, from the workers' perspective, approximately correct on capital taxes, but taxed labor much too highly. Although capital taxes are not much higher, this represents a significant shift in tax revenue towards capital. The lower labor tax rates cause a higher after-tax wage for workers and higher labor supply (8.9%). This in turn raises the return to capital, which causes steady state capital to be 3.6% higher. Output overall is 5.9% higher compared to the competitive outcome in Section 5.1. Note that the lower overall tax rates mean lower gross transfers to workers. However, since capital taxes have gone up slightly, and labor taxes have gone down a lot, transfers to workers net of taxes increases,

Figure 4: Dynamics for Gradual Increase with $\sigma = 1.2$ under majority voting



both absolutely and relative to GDP, compared to the fixed tax case. In terms of gains to workers and entrepreneurs, worker consumption is higher by 7%, and by 1.4% relative to output. Entrepreneur's consumption is 3.8% higher. Thus workers capture more of the gains in output in steady state, though entrepreneurs capture some of them. However, it must be noted that larger consumption gains for workers are offset by increased disutility from supplying labor, whereas there is no such offset for entrepreneurs.

While considering the transition, strikingly, although the policy under majority voting is chosen to maximize worker welfare without reference to entrepreneur welfare at all, and it features higher steady state capital taxes and net transfers to workers, it results in a significantly higher welfare gain from the automation episode for entrepreneurs, but only marginally higher welfare for workers. Specifically, worker welfare increases barely from 24.6% to 26.1% in consumption equivalent terms, whereas for entrepreneurs the increase is much larger: from 57.7% to 84.5%.[17]

At first this might seem strange in light of the steady state numbers. Why would the

[17]The corresponding numbers for $\sigma = 1.2$ are 29.4% versus 30.6% and 49.0% versus 72.5%, which tell a very similar story.

optimal path result in so much higher utility for entrepreneurs, given the steady state tax burden on them is higher? The answer lies in the dynamic paths, which are shown in Figures 3 and 4. Figure 3 shows the result of automation with fixed tax rates, while Figure 4 shows automation with taxes set under majority voting (thus to maximize worker utility). Look at the transfers to workers (lower right panel). Under constant taxes, transfers as a share of GDP rise slowly to their new higher level. Under the optimal policy, they drop sharply on impact, and then rise again slowly over time.

What's going on? Under automation, not only is a large increase in capital optimal from the workers' perspective, but, more importantly, it is optimal for this increase in capital to occur relatively quickly. This is achieved by cutting capital taxes now, which raises entrepreneurs' after tax income and therefore accelerates investment, and hence, results in a quicker increase in the capital stock. While it is better to cut capital taxes initially in order to increase the capital stock more quickly from the workers' perspective, yet this produces only a small, second-order gain in their welfare. The reason is that the resulting benefit to workers in terms of a faster wage increase is offset by a steep initial cut in transfers to them that is required to implement the optimal initial capital tax reduction. By contrast, the initial period of low capital taxes results in a faster increase in entrepreneurs' consumption along with that of their capital (as consumption is a fixed proportion of capital). With no offsetting forces like that in the case of workers, this moving forward in time of entrepreneurial consumption results in a much larger, first-order welfare gain for them than for the workers which is of the second order.

6 Conclusion

Recent trends in automation have re-ignited the debate about the distributional implications of technical progress and possible policy responses. Our analysis in this paper speaks directly to these issues. We find that when the economy is not (yet) fully automated, the effects of automation on wages and labor share of income are worse than that of traditional technical progress. Moreover, if workers have political power, they will choose to tax capital to transfer resources to themselves from the entrepreneurs. Nevertheless, we find that workers will choose to temporarily lower capital taxes during an episode of automation to hasten the accumulation of capital and boost their wages and transfers. The welfare gains from such a policy are much larger for entrepreneurs than for workers, since changes in capital taxes have first-order welfare effects on entrepreneurs, but only second-order welfare effects on workers. This may explain why capital tax rates remain low across advanced economies despite recent concerns about growing inequality and automation, and why recent discussion of "universal basic income" has focused on broadening the scope of transfers rather than raising capital taxes per se. Future research may examine how government's tax and transfer policy may depend on alternative assumptions about the constraints and mechanisms to set such policies. It may also explore the effects of worker heterogeneity on this analysis, since not all workers are equally affected by particular episodes of automation.

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A Omitted Proofs

A.1 Proof of Proposition 1

Proof. We can write the firm optimization problem as

$$\max_{\ell(i),k(i)} \left\{ \left[\int_0^1 \left(a\left(i\right) k\left(i\right) + b\left(i\right) \ell\left(i\right)\right)^{1-\frac{1}{\sigma}} di \right]^{\frac{\sigma}{\sigma-1}} - w \int_0^1 \ell\left(i\right) di - r \int_0^1 k\left(i\right) di \right\}, \tag{65}$$

subject to non-negativity constraints for factor inputs, $k\left(i\right)$ and $\ell\left(i\right)$. This yields two optimality conditions,

$$w \geq b\left(i\right) \cdot \left(\frac{Y}{y\left(i\right)}\right)^{\frac{1}{\sigma}} \tag{66}$$

and

$$r \geq a\left(i\right) \cdot \left(\frac{Y}{y\left(i\right)}\right)^{\frac{1}{\sigma}}, \tag{67}$$

which hold with equality for positive use of labor or capital respectively in the performance of task i.

With linear substitution between capital and labor in the performance of a given task, it will be performed by capital only if $a\left(i\right)/b\left(i\right) > r/w$, and by labor only if $a\left(i\right)/b\left(i\right) < r/w$. If $a\left(i\right)/b\left(i\right) = r/w$, then the task may be performed by both capital or labor. This will often pin down a unique production plan, but if $a\left(i\right)/b\left(i\right) = r/w$ for all tasks in an interval, then there are many equivalent production plans. To avoid this, we adapt the convention that (1) all tasks are done purely by capital or purely by labor, and (2) if $i < j$, then it is never the case that i is done by labor and j by capital. We can then set α such that tasks $i < \alpha$ are done only by capital, and tasks $i > \alpha$ are done only by labor. Therefore, α is the share of tasks done by capital and it will satisfy:

$$\begin{cases} \frac{a(i)}{b(i)} \geq \frac{r}{w} & \text{for } i < \alpha \\ \frac{a(i)}{b(i)} \leq \frac{r}{w} & \text{for } i > \alpha. \end{cases} \tag{68}$$

In the event that $a\left(i\right)/b\left(i\right)$ is strictly decreasing (at least in the neighborhood of α), α will be uniquely defined by

$$\frac{a\left(\alpha\right)}{b\left(\alpha\right)} = \frac{r}{w}. \tag{69}$$

Otherwise, α will be determined together with factor demands.

For tasks done by capital ($i \leq \alpha$), the optimal production of task i satisfies

$$r = \left(\frac{Y}{a\left(i\right) \cdot k\left(i\right)}\right)^{\frac{1}{\sigma}} a\left(i\right). \tag{70}$$

Thus $\left(a\left(i\right)\right)^{1-\sigma} k$ is the same across all $i \leq \alpha$ and aggregate inverse capital demand, given unit measure of entrepreneurs, satisfies:

$$r = \left(A\right)^{\frac{\sigma-1}{\sigma}} \left(\frac{\alpha Y}{K}\right)^{\frac{1}{\sigma}}. \tag{71}$$

where

$$K \equiv \int_0^{\alpha} k\left(i\right) di \tag{72}$$

is aggregate capital and

$$A(\alpha) \equiv \left[\frac{1}{\alpha} \int_0^{\alpha} \left(a\left(i\right)\right)^{\sigma-1} di\right]^{\frac{1}{\sigma-1}} \tag{73}$$

is the productivity of capital. Analogously, inverse labor demand, given a unit measure of workers, satisfies

$$w = \left(B\right)^{\frac{\sigma-1}{\sigma}} \left(\frac{\left(1-\alpha\right) Y}{L}\right)^{\frac{1}{\sigma}}, \tag{74}$$

where

$$L \equiv \int_{\alpha}^{1} \ell\left(i\right) di \tag{75}$$

is aggregate labor and

$$B(\alpha) \equiv \left[\frac{1}{1-\alpha} \int_{\alpha}^{1} \left(b\left(i\right)\right)^{\sigma-1} di\right]^{\frac{1}{\sigma-1}} \tag{76}$$

is labor productivity.

The expressions above allow us to derive the following expression for aggregate production as a function of aggregate capital and labor:

$$Y \equiv F(K,L) = \left[\alpha^{\frac{1}{\sigma}} \left(A(\alpha)K\right)^{1-\frac{1}{\sigma}} + \left(1-\alpha\right)^{\frac{1}{\sigma}} \left(B(\alpha)L\right)^{1-\frac{1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, \tag{77}$$

where $\{A(\alpha), B(\alpha), \alpha\}$ are defined as above and

$$F_L(K,L) = w \tag{78}$$

$$F_K(K,L) = r. \tag{79}$$

The expression for q is just equal to r/w.

□

A.2 Proof of Proposition 2

In steady state, c_e is constant. Then the equations (7) and (8), combined with (6), imply that in steady state:

$$r = \frac{\rho + \delta}{1 - \tau^k} \equiv r^* > 0 \tag{80}$$

Thus the steady-state r, which we call the critical interest rate and denote as r^*, does not depend on technology (besides the depreciation rate). It depends only on the entrepreneurs' time discount rate and the tax rate on capital (plus the depreciation rate).

Thus, while a steady state requires that $r = r^*$, if instead it is always the case that $r > r^*$, then the entrepreneurs' consumption grows continuously, *i.e.*,:

$$\dot{c}_e > 0$$

More generally, there is now continuous growth through capital accumulation. Further, as capital grows, the relative role of labor becomes small, and the economy approaches an AK model. In particular, the labor share of income approaches zero in this case and is zero along the corresponding balanced growth path. This leads us to our first proposition:

First note that, if $r(t) > r^*$, then from the entrepreneur's maximization condition and co-state equations (7, and 8), entrepreneurial consumption c_e is strictly increasing in t, which (from the budget constraint) also implies strictly increasing capital over time, i.e. continuous growth. Thus it is sufficient to prove the claim to show that $r \geq A(1)$.

To show that $r \geq A(1)$, suppose that $L = 0$. In this case, from the aggregate representation of the production function, production satisfies:

$$Y = A(1) \cdot K$$

and the marginal product of capital satisfies:

$$r = A(1)$$

Then the proposition holds as long as r is increasing in L, which we now prove.

We would like to prove that the marginal product of capital (r) is increasing in L. This amounts to proving that the derivative $F_K(K, L)$ is increasing in L, where:

$$F(K, L) = \max_{\alpha} \left\{ \left[\alpha^{\frac{1}{\sigma}} (A(\alpha) \cdot K)^{1-\frac{1}{\sigma}} + (1-\alpha)^{\frac{1}{\sigma}} (B(\alpha) \cdot L)^{1-\frac{1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \right\}$$

We begin by proving the following lemmas.

Lemma 1. *Optimal capital task share α is decreasing in labor/capital ratio L/K.*

Proof. The equilibrium condition that holds at α can be written as:

$$\begin{cases} Z(i) \geq X(\alpha) \cdot \frac{L}{K} & \text{for } i < \alpha \\ Z(i) \leq X(\alpha) \cdot \frac{L}{K} & \text{for } i > \alpha \end{cases}$$

where:

$$X(\alpha) = \frac{\int_0^\alpha (a(i))^{\sigma-1}\, di}{\int_\alpha^1 (b(i))^{\sigma-1}\, di}$$
$$Z(i) = \left[\frac{a(i)}{b(i)}\right]^\sigma$$

(This condition is found by substituting the expressions for r and w into the condition for α, and substituting in the expressions for A and B). By assumption, $Z(i)$ is decreasing in i. Furthermore, since $a(i), b(i) \geq 0$, it is clear that $X(\alpha)$ is increasing in α, since:

$$X'(\alpha) = X(\alpha) \cdot \left(\frac{(a(\alpha))^{\sigma-1}}{\int_0^\alpha (a(i))^{\sigma-1}\, di} + \frac{(b(\alpha))^{\sigma-1}}{\int_\alpha^1 (b(i))^{\sigma-1}\, di} \right) > 0$$

Now suppose that $L_2/K_2 > L_1/K_1$. We show that $\alpha_2 < \alpha_1$ by contradiction. Suppose not, so that $\alpha_2 > \alpha_1$. Take some $i \in (\alpha_1, \alpha_2)$. Since $i > \alpha_1$, it follows that:

$$Z(i) \leq X(\alpha_1) \cdot \frac{L_1}{K_1}$$

and since $i < \alpha_2$, it follows that:

$$Z(i) \geq X(\alpha_2) \cdot \frac{L_2}{K_2}$$

Therefore:

$$X(\alpha_1) \cdot \frac{L_1}{K_1} \geq X(\alpha_2) \cdot \frac{L_2}{K_2}$$

Since by assumption $L_2/K_2 > L_1/K_2$, it follows that:

$$X(\alpha_1) > X(\alpha_2)$$

Since X is increasing in α, this implies that

$$\alpha_1 > \alpha_2$$

which contradicts our assumption. Therefore $\alpha_2 \leq \alpha_1$, which proves the first part of the claim. □

Lemma 2. *The ratio $w/r = F_L(K,L)/F_K(K,L)$ is increasing in $\kappa = K/L$.*

Proof. We know that:

$$\begin{cases} \frac{a(i)}{b(i)} \geq \frac{r}{w} & \text{for } i < \alpha \\ \frac{a(i)}{b(i)} \leq \frac{r}{w} & \text{for } i > \alpha \end{cases}$$

and that α is decreasing in L/K, and therefore increasing in κ. We further know that a/b is decreasing in i.

Now consider what happens if there is an increase in L/K. We can consider two cases. One case is that a/b is continuously decreasing at α. In this case, $r/w = a/b$ holds with equality in the neighborhood of α. Therefore an increase in L/K will cause α to decline, and therefore $a/b = r/w$ will increase.

Conversely, suppose that a/b is discontinuous at i. Then a marginal change in L/K will not change α. But then from:

$$\frac{r}{w} = \left(\frac{\int_0^\alpha (a(i))^{\sigma-1}\,di}{\int_\alpha^1 (b(i))^{\sigma-1}\,di}\right)^{\frac{1}{\sigma}} \left(\frac{L}{K}\right)^{\frac{1}{\sigma}}$$

we immediately get that r/w must increase.

Thus in either case a marginal increase in L/K causes a marginal increase in r/w. □

Now we are in position to prove our final lemma:

Lemma 3. *The marginal product of capital $F_K(K,L)$ is increasing in L, since $F_{KL}(K,L) > 0$.*

Proof. We can rewrite the production function as:

$$F(K,L) = \max_\alpha \left\{ \left(\alpha^{\frac{1}{\sigma}} (A)^{\frac{\sigma-1}{\sigma}} + (1-\alpha)^{\frac{1}{\sigma}} \left(B\frac{L}{K} \right)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} K \right\}$$

This implies that the choice of α that maximizes production for a given (K,L) depends only on the ratio $\kappa = K/L$.

Next we calculate the derivatives of the aggregate production function. These are:

$$F_K(K,L) = \left(\frac{\alpha A^{\sigma-1}Y}{K}\right)^{\frac{1}{\sigma}} = \left(\alpha A^{\sigma-1}\right)^{\frac{1}{\sigma}} \left(\alpha^{\frac{1}{\sigma}}(A)^{\frac{\sigma-1}{\sigma}} + (1-\alpha)^{\frac{1}{\sigma}}\left(B\frac{L}{K}\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}}$$

$$F_L(K,L) = \left(\frac{(1-\alpha)B^{\sigma-1}Y}{L}\right)^{\frac{1}{\sigma}} = \left((1-\alpha)B^{\sigma-1}\right)^{\frac{1}{\sigma}} \left(\alpha^{\frac{1}{\sigma}}\left(A\frac{K}{L}\right)^{\frac{\sigma-1}{\sigma}} + (1-\alpha)^{\frac{1}{\sigma}}(B)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}}$$

The key thing to observe is that both equations depend only on $\kappa = K/L$, and not on K or L independently. Therefore we can write each as:

$$F_K(K,L) = r(\kappa)$$
$$F_L(K,L) = w(\kappa)$$

Moreover, from Lemma 2 we know that:

$$\frac{w(\kappa)}{r(\kappa)} = \frac{F_L(K,L)}{F_K(K,L)} = \left(\frac{(1-\alpha)B^{\sigma-1}}{\alpha A^{\sigma-1}}\kappa\right)^{\frac{1}{\sigma}}$$

is increasing in κ.

Now we can prove the claim. Note that changes in L and K affect F_K and F_L only by altering κ. Therefore we have:

$$F_{KL} = \frac{d}{dL}(F_K) = -r'(\kappa)\cdot\frac{K}{L^2}$$
$$= \frac{d}{dK}(F_L) = w'(\kappa)\cdot\frac{1}{L}$$

Therefore:

$$-r'(\kappa)\cdot\kappa = w'(\kappa)$$

This implies that $w'(\kappa)$ and $r'(\kappa)$ have opposite signs.

In Lemma 2 we showed that w/r is increasing in κ. Therefore:

$$\frac{d}{d\kappa}\left(\frac{w(\kappa)}{r(\kappa)}\right) = \frac{w}{r}\left(\frac{w_\kappa}{w} - \frac{r_\kappa}{r}\right) \geq 0$$

Now, since r_κ and w_κ have opposite signs, the only way that:

$$\frac{w_\kappa}{w} \geq \frac{r_\kappa}{r}$$

is if $w_\kappa \geq 0$ and $r_\kappa \leq 0$. From this we conclude that $r_\kappa \leq 0$, and therefore:

$$F_{KL} = -r'(\kappa) \cdot \frac{K}{L^2} \geq 0$$

That is, the return on capital is increasing in L/K, and therefore in L. □

A.3 Proofs from Section 3.2

We begin with the following lemma:

Lemma 4. *Given that (25) holds and $b(i) = 1$, and given a value of $L > 0$, there exists a $\bar{K}(a, \bar{\alpha}) = \frac{\bar{\alpha}}{1-\bar{\alpha}} \frac{L}{a}$ such that for $0 \leq K \leq \bar{K}(a, \bar{\alpha})$, the production function has following linear representation*

$$Y(K, L) = aK + L. \tag{81}$$

Proof. Using the expressions from Proposition 1, note that $a(i)/b(i) = a$ for $\alpha \leq \bar{\alpha}$. Thus the $a(i)/b(i)$ curve is constant over this region, and therefore at any interior point $\alpha < \bar{\alpha}$, condition (15) implies that $q = a$. Further, for $\alpha < \bar{\alpha}$, we have:

$$q = \left(\frac{\alpha}{1-\alpha} a^{\sigma-1} \frac{L}{K}\right)^{\frac{1}{\sigma}}$$

and therefore $q = a$ implies:

$$\alpha = \frac{aK}{aK + L}$$

Plugging this into the production function (12), with $A = a$ and $B = 1$, yields:

$$Y = \left(\left(\frac{aK}{aK+L}\right)^{\frac{1}{\sigma}} (aK)^{1-\frac{1}{\sigma}} + \left(\frac{L}{aK+L}\right)^{\frac{1}{\sigma}} L^{1-\frac{1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} = aK + L$$

This holds as long as $\alpha < \bar{\alpha}$, which using the expression for α above, is equivalent to:

$$K < \frac{\bar{\alpha}}{1-\bar{\alpha}} \frac{L}{a} \equiv \bar{K}$$

Meanwhile, when $K = \bar{K}$ exactly, we have $\alpha = \bar{\alpha}$, and therefore $Y = aK + L$ still holds. □

Intuitively, when capital is below a threshold value, there is insufficient capital to perform all tasks that capital is capable of performing, and thus labor and capital are perfectly substitutable at the margin.[18]

[18]The linear representation of production in Lemma 4 has a strong resemblance to the "cone of diversifi-

We now turn to characterizing what happens in steady state. The following lemma summarize the possibilities, depending on initial parameters:

Lemma 5. *In steady state, (i) if $a < r^*$, then $\alpha = 0$ and equilibrium capital stock $K = 0$ implying $Y(K, L) = Y(L) = L$, (ii) if $a = r^*$, then there is a continuum of steady states with $\alpha \in [0, \bar{\alpha}]$ and equilibrium capital stock $K = \bar{K}(r^*, \alpha)$ so that $Y(K, L) = Y(\bar{K}, L) = a\bar{K}(r^*, \alpha) + L$, and (iii) if $a > r^*$, then $\alpha = \bar{\alpha}$ and equilibrium capital stock $K = \hat{K}(a, \bar{\alpha}, r^*) = \frac{\bar{\alpha}}{(1-\bar{\alpha})^{-\frac{1}{\sigma-1}}((\frac{r^*}{a})^{\sigma-1}-\bar{\alpha})^{\frac{\sigma}{\sigma-1}}} \frac{L}{a} > \bar{K}(r^*, \bar{\alpha})$ with $Y(K, L) = \left((\bar{\alpha})^{\frac{1}{\sigma}} (aK)^{\frac{\sigma-1}{\sigma}} + (1-\bar{\alpha})^{\frac{1}{\sigma}} L^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$ so the production function exhibits the usual diminishing returns in both factors.*

Proof. The results follow easily from the linear representation in Lemma 4. (i) follows from the fact that the return on capital (a) is below the return that the entrepreneurs want (r^*); (ii) arises because in the knife edge case of $a = r^*$ any value of $0 \leq K \leq \bar{K}(r^*, \bar{\alpha})$ is consistent with the entrepreneurs' Euler equation in the steady state because with a linear production function there are no diminishing returns to capital accumulation in that range of K; and (iii) is the case where Lemma 4 is not applicable so the argument in (ii) about no diminishing returns does not apply.

To derive the expression for $\hat{K}$, first note that when $a > r^*$, we know that $\alpha = \bar{\alpha}$ binds in steady state. Therefore in steady state we have:

$$F_K = a\left(\frac{\alpha Y}{aK}\right)^{1/\sigma} = r^*$$

where production is:

$$Y = \left(\bar{\alpha}^{1/\sigma} (aK)^{1-1/\sigma} + (1-\bar{\alpha})^{1/\sigma} L^{1-1/\sigma}\right)^{\frac{\sigma}{\sigma-1}}$$

Combining these and solving for K yields:

$$K = \frac{\bar{\alpha}\,(1-\bar{\alpha})^{\frac{1}{\sigma-1}}}{\left[(r^*/a)^{\sigma-1} - \bar{\alpha}\right]^{\frac{\sigma}{\sigma-1}}} \frac{L}{a}$$

which is the expression for $\hat{K}$ given above. Note that $\hat{K}(r^*, \bar{\alpha}, r^*) = \bar{K}(r^*, \bar{\alpha})$. Finally, to see that $K > 0$, recall that we assumed $a > r^*$ and $r^* > A(1)$. When $\sigma < 1$, the first implies the

cation" for production in a small-open economy (see Dixit and Norman (1980), Bhagwati et al. (1998), and Grossman and Helpman (1991)). For example, in a two-good, two-factor economy with goods prices fixed in international markets, zero profit conditions for the production of the two goods imply fixed values of the two factor prices. Thus, within the cone of diversification, the aggregate economy displays no diminishing returns to changes in (relative) quantity (or endowment) of factors just as our linear representation in (81).

denominator is positive. When $\sigma > 1$, the second, together with $A(1) = a\,(\bar{\alpha})^{\frac{1}{\sigma-1}}$, implies the denominator is positive. $\square$

Now it is fairly straightforward to prove Proposition 3:

Proof of Proposition 3. Given assumption (26), we are in case (iii) of Lemma 5. The expression for K/L therefore follows immediately from the expression for $\hat{K}$. The expression from output follows from the expressions in Proposition 1, with $\alpha = \bar{\alpha}$, $A(\alpha) = a$, and $B = 1$. Output is positive since $K > 0$. The wage satisfies $w = F_L = ((1-\alpha)Y/L)^{1/\sigma}$, which given the expression for $\hat{K}$, has the given expression. Finally, the labor share is

$$s_L = \frac{wL}{Y} = (1-\alpha)^{1/\sigma}\,(Y/L)^{1/\sigma-1}$$

which, given the expressions for Y and K/L, yields the given expression for the labor share. Both are positive because $K > 0$ and $Y > 0$. $\square$

The results of corollaries 2 and 3 follow immediately from the expressions for the wage and labor share:

Proof of Corollary 2. Differentiating the expression for steady state wage (29) and labor share (30) yields equations (31) and (32). Note that the denominator of (31) is positive by the same argument that $K > 0$. $\square$

Proof of Corollary 3. Follows directly from differentiating equations (29) and (30). The expression in (33) is positive because $(r^*/a)^{\sigma-1} > \bar{\alpha}$, by the same argument as for $K > 0$, and since $a > r^*$, $1 - (r^*/a)^{\sigma-1}$ has the same sign as $\sigma - 1$. $\square$

A.4 Proof of Proposition 4

Proof. When $\alpha = \bar{\alpha}$, output is now:

$$Y = \left(\bar{\alpha}^{1/\sigma}\,(aK)^{1-1/\sigma} + (1-\bar{\alpha})^{1/\sigma}\,(BL)^{1-1/\sigma}\right)^{\frac{\sigma}{\sigma-1}}$$

where now $B \neq 1$ is possible. Since $r = r^*$ and $\alpha = \bar{\alpha}$ hold in steady state, the steady state effective capital-labor ratio is still:

$$\frac{aK}{BL} = \frac{\bar{\alpha}\,(1-\bar{\alpha})^{\frac{1}{\sigma-1}}}{\left[(r^*/a)^{\sigma-1} - \bar{\alpha}\right]^{\frac{\sigma}{\sigma-1}}}$$

Then the steady state wage can be written as:

$$w^* = B \left(\frac{1 - \bar{\alpha} \left(r^*/a \right)^{1-\sigma}}{1 - \bar{\alpha}} \right)^{\frac{1}{1-\sigma}}$$

Let w_0 be the steady state wage initially, and let w_1 be the steady state wage after the episode of automation. Before automation, we have:

$$B_0 = \left(\frac{\bar{\alpha}_1 - \bar{\alpha}_0}{1 - \bar{\alpha}_0} b_m^{\sigma-1} + \frac{1 - \bar{\alpha}_1}{1 - \bar{\alpha}_0} b_1^{\sigma-1} \right)^{\frac{1}{\sigma-1}}$$

and after automation we have $B_1 = b_1$. Using these expressions, the ratio of w_1 to w_0 satisfies:

$$\left(\frac{w_1}{w_0} \right)^{1-\sigma} = \frac{1 - \bar{\alpha}_1 \left(\frac{r^*}{a} \right)^{1-\sigma}}{1 - \bar{\alpha}_0 \left(\frac{r^*}{a} \right)^{1-\sigma}} \left\{ 1 + \left(\frac{\bar{\alpha}_1 - \bar{\alpha}_0}{1 - \bar{\alpha}_1} \right) \left(\frac{b_1}{b_m} \right)^{1-\sigma} \right\} \tag{82}$$

We are interested in knowing when the steady state wage declines following automation, i.e. when $w_0 < w_1$. From (82), we can work out that this will happen when condition (36) holds.

□