Network Contest Games

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Abstract

This paper considers the impact of identity-dependent externalities on competitive behavior in all-pay contests. We introduce a model of *network contest games*, in which the prize generates payoff externalities for players directly linked to the winner, and establish existence and sufficient conditions for uniqueness of Nash equilibria for arbitrary networks with heterogeneous links. Our uniqueness result provides a novel adaptation of well-known results for network games with linear best reply functions to the network contest game, which features non-linear best replies. We then provide specific characterizations and illustrations of equilibria for several tractable cases, including networks with homogenous links, and networks with heterogeneous links, but homogenous node strengths. Variations in the network structure and the nature of the externalities have intuitive consequences for equilibrium investment. In general, the presence of positive externalities introduces free-riding incentives, whereas negative externalities intensify competition, especially among highly connected agents.

Keywords: contests, networks, identity-dependent externalities, network games, best-response potential

JEL: C72, C92, D72, D74, D85, Z13

1 Introduction

In virtually all areas of social and economic interaction, one can find examples of agents competing with each other in pursuit of some valuable prize. Individuals and organizations frequently expend significant resources on marketing, advertising, and lobbying in order to outperform their rivals or command a greater

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influence over market allocations or political outcomes. Research in industrial economics, public choice, and political economy has explored competitive behavior in rent-seeking environments, R&D competition, patent races, political campaigns, and promotion tournaments. Many of these settings are modeled as *contests*, in which agents exert costly effort or make irreversible investments and the winner takes all.

The standard assumption in contest models is that losing agents are indifferent to the identity of the winner. However, agents may have considerably more general preferences over the possible allocations of the prize. In particular, agents who do not win the contest may care a great deal about who does, especially if the allocation of the prize affects the nature of subsequent interactions between the contestants. In the related context of winner-pay auctions, Jehiel, Moldovanu and Stacchetti (1996) introduced the notion of *identity-dependent externalities* (or IDEs) as a way of capturing the consequences of the allocation for bidders in post-auction interactions. Such externalities may arise with the assignment of exclusive licensing agreements (Brocas, 2003), the sale of a nuclear weapon or location of environmentally hazardous enterprises (Jehiel, Moldovanu and Stacchetti, 1996), competition for access to a cost-reducing process innovation, or the allocation of talent across teams (Das Varma, 2002).

There are relatively few studies that consider the implications of IDEs for *all-pay contests* (see, e.g., Linster, 1993; Esteban and Ray, 1999; Konrad, 2006; Klose and Kovenock, 2015); and yet, there remain many interesting questions to explore. For instance, in many settings, the structure of IDEs is governed by an underlying network of connections. As such, there are naturally arising questions regarding the impact of network structure on competitive behavior which, to date, have not been addressed by the existing literature on IDEs in auctions and contests.

In this paper, we study the effects of network-based identity-dependent externalities on competitive behavior in all-pay contest environments. To do so, we develop and analyze a theoretical model of a *network contest game*. Our framework builds on recent developments in the understanding of strategic behavior in games played on networks (Bramoullé, Kranton and D'Amours, 2014). We concentrate on Tullock (1980) contests—one of the most commonly studied formulations of *imperfectly-discriminating* all-pay contests—wherein each player's probability of winning the contest is increasing in her own effort investment, relative to the investments of others. The primary innovation of our model is the introduction of a network that governs the flow of externalities from the winning player to her neighbors.

As a motivating example, consider a collection of community councils lobbying a city planning committee in charge of selecting the location for a new public facility. Each community's ideal outcome would be to have the facility located within its own neighborhood. However, if the facility generates positive externalities or is more easily accessible to neighborhoods that are sufficiently close to the eventual location, it is natural to expect that lobbying activity will depend on the geographical network connecting the communities. If the externalities are sufficiently strong, or the communities sufficiently well-connected, they may engage in less lobbying activity than if it is more difficult to access a facility located outside their own neighborhood.

Along similar lines, the investment decisions made by firms competing for an exclusive licensing agreement will typically depend on the rivalry structure in the firms' product market space. Firms that operate in close proximity to the winning firm may be significantly worse off than other unsuccessful firms.[1] How might the structure of product market rivalries affect rent-seeking behavior in this setting? The natural intuition in this case suggests that the negative externalities associated with the exclusive license will intensify competition among firms who are engaged in markets with more heated rivalry.

Our main contributions in this paper are as follows. First, we establish the existence of a Nash equilibrium for general network structures and externalities (Theorem 1). The main challenge to existence is the fact that payoff functions in the network contest game are (like the standard contest environment) discontinuous at zero. We rely on results from Reny (1999) and Bagh and Jofre (2006) to prove existence. Second, we provide sufficient conditions for there to be a unique equilibrium (Theorem 2). Our characterization closely follows the seminal approach developed by Bramoullé, Kranton and D'Amours (2014) for network games with linear best replies. However, adapting their results to the network contest game is a non-trivial exercise. In particular, because best replies are non-

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[1] Another similar example can be made in the context of professional sporting organizations competing for the services of a talented free-agent athlete. For instance, in Major League Baseball, the Boston Red Sox (part of the American League East Division) might be much happier to see a top free agent player sign a deal with the San Diego Padres (who are in the National League West Division) than with the New York Yankees, who play in the same League and Division as Boston. There are, of course, several other considerations that influence the negotiations between sporting teams and free agent athletes, including salary demands, team budgets, contract length, synergies with existing team members, and the athlete's locational preferences. Nevertheless, the point is that competition in these kinds of settings, which may include both winner-pay and all-pay components, is likely influenced by the anticipated interest and activity of rival teams.

linear in the network contest game (just as they are for standard Tullock (1980) contests), the main results derived by Bramoullé, Kranton and D'Amours (2014) cannot be directly applied.[2] Nevertheless, we demonstrate that the key insights provided by Bramoullé, Kranton and D'Amours (2014) can be suitably adapted to the network contest game.

One key condition of our uniqueness theorem relates the size of the externalities in the network contest game to the lowest eigenvalue of the network, which also plays a crucial role in Bramoullé, Kranton and D'Amours (2014).[3] While Bramoullé, Kranton and D'Amours (2014) exploit the theory of potential games (Monderer and Shapley, 1996) to derive their results, our formulation does not admit an exact potential function. Instead, we establish that the network contest game is a *best-response potential game* (Voorneveld, 2000), which allows us to take an analogous approach. The uniqueness theorem can be broken into two cases. On the one hand, when all externalities are negative (or zero), the lowest-eigenvalue condition is sufficient on its own. On the other hand, if there are any strictly positive externality flows, we provide, via direct argument, two other conditions that, when satisfied together with the lowest-eigenvalue condition, are sufficient for there to be a unique equilibrium. Altogether, our theoretical framework establishes new results extending both the well-developed literature on contest theory and the growing body of work studying strategic behavior in network games.

In addition to our fully general results, we provide closed-form characterizations and illustrations of equilibria for two more tractable settings. The first is the restricted setting in which links are homogenous—meaning all externality flows are identical in sign and magnitude.[4] For this case, we examine two broad classes of network structures: *regular* networks and (a subclass of) *core-periphery* networks, to highlight key characteristics of the relationship between externalities, network properties, and equilibrium behavior. For regular networks, there exists a symmetric equilibrium in any network contest game. Moreover, comparative statics with respect to the size of the externality and the density of the network are con-

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[2] Moreover, approaches based on variational inequalities (VI) that have been applied to network games without linear best replies (see, e.g., Melo, 2018; Parise and Ozdaglar, 2019; Zenou and Zhou, 2020) also do not apply.

[3] As discussed by Bramoullé, Kranton and D'Amours (2014), the lowest eigenvalue captures the "two-sidedness" or "bipartiteness" of the graph. When the lowest eigenvalue (which is negative) is sufficiently large in magnitude, the amplification of agents' interactions increases the chances of multiple equilibria.

[4] In a companion paper, we report the results of a controlled laboratory experiment designed to test several key comparative statics in network contest games with homogenous links; see Boosey and Brown (2022).

sistent with the intuition highlighted by the motivating examples given above. For instance, positive externalities introduce incentives for players to free-ride on their neighbors' investments, leading to lower equilibrium investment. Conversely, negative externalities drive up the effective value of winning the contest, intensifying competition and increasing equilibrium investment. Each of these effects is amplified as the network becomes more densely connected, as captured by an increase in the common *degree* for regular networks. Nevertheless, the symmetric equilibrium in regular networks is typically not unique. For instance, when externalities are positive and sufficiently strong, there may also exist a *specialized equilibrium*, in which some subset of the players choose to be inactive (invest nothing) in the contest.

Similarly, semi-symmetric equilibria in a subclass of core-periphery networks also take the form of a specialized equilibrium for sufficiently strong, positive externalities. In particular, highly connected core players, facing stronger free-riding incentives than peripheral players, invest nothing in equilibrium. In contrast, when the prize allocation generates strong negative externalities, the core players—who are more exposed by the structure of the network—increase their equilibrium investment substantially compared to the peripheral players.

The second setting we examine allows for some heterogeneity across links but retains homogeneity with respect to node strength. That is, links may have different weights in the network, but for each player, the sum of the weights on all of their links is the same. We provide several examples to distinguish this setting from the homogenous links case, then relate our model to a network contest game between competing alliances, and a canonical formulation of (standard) group contests with a public-good prize. Finally, to provide some illustration of the model's generality, we introduce two examples (using similar network structures as for the other two sections) in which there are both heterogeneous links and heterogeneous node strengths.

Related literature.— Our study contributes to and draws together two separate literatures. The first of these explores the implications of identity-dependent externalities for strategic behavior in competitive environments. The second is the relatively more recent literature studying games played on networks. In addition, our work naturally relates to the vast body of research on contests.

Prior literature on IDEs, following Jehiel, Moldovanu and Stacchetti (1996) has mostly considered optimal selling procedures in the presence of identity-dependent externalities.[5] Other related work has focused on strategic non-participation in auctions, especially with negative externalities (see, e.g., Jehiel and Moldovanu, 1996; Brocas, 2003) and explored the notion of *type-dependent* externalities (Brocas, 2013*a*, 2014), according to which the externality flows are correlated with the players' private valuations and not just their identities. In all-pay contest environments, there are a handful of related studies, including Konrad (2006) and Klose and Kovenock (2015), both of which characterize equilibria in the context of (perfectly-discriminating) all-pay auctions. There are, however, relatively few studies that consider externalities in the context of *imperfectly-discriminating* all-pay contests.

One exception is Linster (1993), who analyzes the equilibrium of a generalized Tullock contest in which the players care about who wins the prize if they do not. Another exception is Esteban and Ray (1999), which explores the relationship between equilibrium conflict and the distribution of preferences over outcomes in a lottery contest between interest groups.[6] While both of these studies incorporate the notion of identity-dependent externalities into a Tullock-style contest, neither draws a formal connection between these externalities and the underlying network structure that governs them. In contrast, a key contribution of our study is to bring together the literature on identity-dependent externalities and the relatively more recent developments in the theory of network games.

Typically, the network games literature examines games with linear best replies (see, e.g., the linear-quadratic utility functions in Ballester, Calvó-Armengol and Zenou, 2006; Bramoullé and Kranton, 2007; Bramoullé, Kranton and D'Amours, 2014). Among those that consider games with non-linear best replies, Allouch (2015) studies the private provision of local (network-based) public goods, and Melo (2018), Parise and Ozdaglar (2019), and Zenou and Zhou (2020) apply techniques based on variational inequalities (VI) to establish existence and uniqueness.[7] To the best of our knowledge, the only other study to forge the connection

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[5] For instance, Jehiel, Moldovanu and Stacchetti (1996, 1999) characterize the revenue-maximizing auctions for alternative information structures (including the case where externality flows are private information), Jehiel and Moldovanu (2000) study efficient auction design with externalities, while Das Varma (2002) characterizes the revenue and efficiency rankings of the standard sealed-bid and open ascending bid auction formats. See Jehiel and Moldovanu (2006) for a summary of the literature on standard, winner-pay auctions with identity-dependent externalities. In addition, Lu (2006) and Brocas (2013*b*) extend the analysis of the optimal auction to include the possibility of externalities between the seller and the bidders, whereas Aseff and Chade (2008) derive the optimal mechanism for a seller with multiple identical units.

[6] A crucial aspect of their model is the introduction of a "metric" over the different groups, which allows for spatial preferences over the preferred outcomes of other interest groups.

[7] Our model also entails non-linear best replies. However, as noted above, the VI approaches

between network games and externalities in a lottery contest game is König et al. (2017). They develop a stylized model of conflict to capture the impact of informal networks of alliances and enmities on conflict expenditures and outcomes, then apply their model to study empirically the Second Congo War.[8] In their model, agents (or groups) compete for a divisible prize in which any group's share of the prize depends on the group's relative *operational performance*, which takes the form of a generalized Tullock CSF. However, in contrast with our model, there are no allocation-based spillovers in their setting.[9]

The remainder of the paper is organized as follows. In Section 2, we introduce the theoretical model of a network contest game. Section 3 presents the equilibrium analysis, including our main results on the existence and uniqueness of Nash equilibria in the network contest game. Specific results for the case in which links are all homogenous are provided in Section 4. We present several illustrations for the case with heterogeneous links but homogenous node strengths in Section 5. Finally, we present two examples with general heterogeneity in Section 6 and provide brief concluding remarks in Section 7.

2 The Network Contest Game

Consider the environment with a set of players $N = \{1, \ldots, n\}$ and a weighted network $\mathbf{G}$, where $g_{ij} \in \mathbb{R}$ represents the weight on a link between two agents i and j. We assume the network is undirected, such that $g_{ij} = g_{ji}$ and adopt the convention that $g_{ii} = 0$ for all $i \in N$.

Each individual competes in a contest by choosing a level of investment (or effort) $x_i \geqslant 0$. All players have the same linear cost of effort function, $c(x_i) = x_i$. Let $\mathbf{x}_{-i}$ denote the vector of investments chosen by all individuals other than i and suppose the probability of player i winning the contest is given by the Tullock

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adopted by Melo (2018) and Parise and Ozdaglar (2019) rely on an assumption that the objective function for each agent depends only on her own action and a neighborhood aggregate, which is not satisfied in our contest game due to the dependence of the contest success function on all players' actions.

[8] There is also a related, though distinct literature on *conflict networks* (see, e.g., Goyal and Vigier, 2014; Franke and Öztürk, 2015; Matros and Rietzke, 2018; Kovenock and Roberson, 2018; Xu, Zenou and Zhou, 2019) and the formation of conflict networks (Hiller, 2017; Jackson and Nei, 2015). In contrast with both our model and the model in König et al. (2017), these studies typically focus on environments where the network is used to describe the structure of conflict between agents who participate in *multiple battles*.

[9] Instead, the effort investments of other groups in König et al. (2017) feed directly into each group's operational performance through the underlying network of alliances and enmities.

(1980) lottery contest success function. That is,

$$P_i(x_i, \mathbf{x}_{-i}) = \begin{cases} \frac{1}{n}, & \text{if } \sum_{h=1}^{n} x_h = 0, \\ \frac{x_i}{\sum_{h=1}^{n} x_h}, & \text{otherwise.} \end{cases} \qquad [1]$$

The winner of the contest receives a prize $V > 0$. We assume, without loss of generality, that the value of the prize is normalized to $V = 1$. In the standard contest setting, player i's payoff from winning is $V = 1$, while the payoff from losing is zero, regardless of who among the other players wins the contest. In such a setting, it is a well-known result (see, e.g., Szidarovszky and Okuguchi, 1997) that the unique equilibrium is symmetric, given by $x_i = \bar{x}$ for all $i = 1, \ldots, n$, where

$$\bar{x} = \frac{n-1}{n^2} \qquad [2]$$

The main innovation in our model is that there are identity-dependent externalities generated by the allocation of the prize that, together with the network, lead to different possible payoffs for player i when she does not win the contest.

In particular, if a player does not win the contest, her payoff depends on whether or not she is linked to the winner, and if so, on the weight of the link between them. The allocation of the prize to a player i imposes an externality g_{ij} on each other agent j. As is natural, if $g_{ij} = 0$, then no externality is imposed on player j. We make the following assumption on the magnitude of the externalities.

Assumption 1. All externality flows are strictly smaller (in magnitude) than the value of the prize, $V = 1$.

Assumption 1 has the appealing feature that it ensures a player never prefers to lose the contest than to win it, holding fixed her level of investment.[10] In order to facilitate Assumption 1, it will be convenient to normalize the link weights such that externalities are given by αg_{ij}, where $\alpha \in [0, 1)$ and $g_{ij} \in [-1, 1]$ for all $i, j \in N$. For instance, suppose the true link weights are given by $h_{ij} \in (-1, 1)$. Let $\alpha = \max_{\{i,j\}} |h_{ij}|$ and define the normalized link weights by $g_{ij} = h_{ij}/\alpha$, provided $\alpha \neq 0$.[11] Throughout the paper, we therefore assume from the outset that $g_{ij} \in [-1, 1]$ for all $i, j \in N$, and that the externality imposed on player j when player

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[10] Strictly speaking, we could allow negative externalities that are larger in magnitude than the prize, and require only that positive externalities are no greater than V.

[11] Note that $\alpha = 0$ can only occur if all link weights are zero, i.e., only if the network is the empty network. Thus, when $\alpha = 0$, simply let $g_{ij} = h_{ij} = 0$.

i wins the contest is given by the term αg_{ij} with $\alpha \in [0, 1)$.[12] It follows that the expected payoff to player i from a profile of investments $(x_i, \mathbf{x}_{-i})$ can be written as

$$\pi_i(x_i, \mathbf{x}_{-i}; \alpha, \mathbf{G}) = P_i(x_i, \mathbf{x}_{-i}) - x_i + \alpha \sum_{j=1}^{n} g_{ij} P_j(x_j, \mathbf{x}_{-j}). \quad [3]$$

Hereafter, we refer to the game described above as a *network contest game*, represented in normal form as $\Gamma = (X_i, \pi_i)_{i=1}^{n}$ where $X_i = \mathbb{R}_+$ represents the strategy set for player i, and $\pi_i(\cdot)$ is the payoff function defined in [3].

3 Equilibrium Analysis

3.1 Existence and Uniqueness

Existence of equilibrium.—For a given profile $\mathbf{x}$, we denote the set of active agents (those for whom $x_i > 0$) by A and the set of inactive agents by $N - A$. We start our analysis by noting that any strategy profile with only one active agent cannot be a Nash equilibrium. Indeed, for a strategy profile $\mathbf{x}$ with $x_j > 0$ and $\mathbf{x}_{-j} = \mathbf{0}$, player j's best response function is empty. Similarly, given $\alpha < 1$, it is also straightforward to show that $\mathbf{x} = \mathbf{0}$ is not an equilibrium. Thus, we can restrict attention to strategy profiles with at least two active agents.

Consider player i and fix a profile $\mathbf{x}_{-i}$ with at least one strictly positive investment. The expected payoff for player i in equation [3] can be rewritten as

$$\pi_i(x_i, \mathbf{x}_{-i}; \mathbf{G}) = \frac{x_i}{\sum_{h=1}^{n} x_h} - x_i + \alpha \sum_{j=1}^{n} g_{ij} \frac{x_j}{\sum_{h=1}^{n} x_h}$$

for all $x_i \geqslant 0$ and all $\mathbf{x}_{-i} \neq \mathbf{0}$. Note that $\partial^2 \pi_i / \partial x_i^2 < 0$ so that the payoff functions are strictly concave. Thus, player i's best response to $\mathbf{x}_{-i} \neq \mathbf{0}$ is a well-defined, single-valued function given by

$$f_i(\mathbf{x}_{-i}; \alpha, G) = \max \left\{ 0, \left[\sum_{h \neq i} x_h (1 - \alpha g_{ih}) \right]^{0.5} - \sum_{h \neq i} x_h \right\}. \quad [4]$$

As in the standard contest game, the best response functions are non-linear. As such, the main analysis of uniqueness and stability for network games developed in Bramoullé, Kranton and D'Amours (2014) cannot be directly applied. More-

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[12] This normalization also implies that $\max_{\{i,j\}} |g_{ij}| = 1$. That is, the strongest link weight is equal to 1 in magnitude. Assumption 1 is then guaranteed by noting that $\alpha < 1$.

over, the payoff functions do not satisfy the assumptions on the objective function required to apply the variational inequalities approach followed by Parise and Ozdaglar (2019) and Melo (2018) for network games with non-linear best replies.[13] When $\alpha = 0$, the best response functions are, as expected, the same as those for the standard contest game, for which existence and uniqueness are well established. For $\alpha \neq 0$, the issue is not quite as straightforward. To prove the existence of a pure strategy Nash equilibrium, we rely on results from Reny (1999) and Bagh and Jofre (2006), to deal with the fact that payoff functions are discontinuous at $\mathbf{x} = 0$.

Theorem 1 (Existence)**.** *The network contest game possesses a pure strategy Nash equilibrium.*

Here, we highlight the main idea behind the proof of Theorem 1, which is detailed along with all of the other proofs in Appendix A. In particular, existence follows from Theorem 3.1 in Reny (1999). In order to apply Reny's theorem, we establish that the network contest game is compact, quasi-concave, and better-reply secure. For the last property, we show that the game is payoff secure and *weakly reciprocal upper semicontinuous* (wrusc), which is a condition introduced by Bagh and Jofre (2006) who prove that payoff security and wrusc imply better-reply security.

Next, we provide a characterization of equilibrium profiles. The following lemma provides a straightforward characterization of the set of Nash equilibria for the network contest game with network $\mathbf{G}$ and $\alpha \in [0, 1)$.

Lemma 1. *An investment profile* $\mathbf{x}$ *with active agents* A *is a Nash equilibrium if and only if* $|A| \geqslant 2$ *and*

(i) *for all* $i \in A$,

$$\sum_{j \in A} (1 - \alpha g_{ij}) x_j - x_i = \left(\sum_{j \in A} x_j \right)^2 \qquad [5]$$

(ii) *for all* $i \in N - A$,

$$\sum_{j \in A} (1 - \alpha g_{ij}) x_j \leqslant \left(\sum_{j \in A} x_j \right)^2 \qquad [6]$$

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[13] They each consider games in which the objective function depends on x_i and a neighborhood aggregate, $\sum_h g_{ih} x_h$, but does not depend otherwise on x_j if $g_{ij} = 0$. In our setting, the payoff of an agent i depends on each x_j through the CSF, even if $g_{ij} = 0$.

Lemma 1 is particularly useful when it comes to constructing closed-form expressions for equilibria in certain special cases of the more general environment. We introduce several of these in Sections 4–6.

Uniqueness of equilibrium.—We turn next to the question of uniqueness. Since the game does not admit linear best replies, we cannot directly apply the results from Bramoullé, Kranton and D'Amours (2014) in order to characterize a sufficient condition for uniqueness. However, using a similar approach, combined with direct argument, we are able to provide a related characterization of sufficient conditions under which the network contest game possesses a unique equilibrium.

To facilitate the exposition, we provide a general description of our approach. First, we show that while the contest game with network externalities is not an exact potential game, it is a *best-response (or best-reply) potential game* (Voorneveld, 2000). That is, there exists a function $\mathbf{P}$ (called a BR-potential) with the same best replies as the network contest game. Thus, the set of Nash equilibria in the game coincides with those strategy profiles that maximize the BR-potential, $\mathbf{P}$.

Second, we partition the domain $\mathbf{X}$ of the BR-potential $\mathbf{P}$ into two subsets: $\mathbf{X}^H$, consisting of strategy profiles $\mathbf{x}$ such that $\sum_h x_h \geqslant 0.5$, and $\mathbf{X}^L$, consisting of strategy profiles $\mathbf{x}$ such that $\sum_h x_h < 0.5$. For $\mathbf{X}^H$, the BR-potential $\mathbf{P}$ is strictly concave in $\mathbf{x}$ as long as $[\mathbf{I} + \alpha\mathbf{G}]$ is positive definite, which is true if and only if $\alpha < 1/|\lambda_{\min}(\mathbf{G})|$, where $\lambda_{\min}(\mathbf{G})$ is the lowest eigenvalue of $\mathbf{G}$. This is the familiar sufficient condition provided by Bramoullé, Kranton and D'Amours (2014) for uniqueness in network games with linear best replies.

When all links are negative ($g_{ij} \leqslant 0$ for all i, j), we show directly that any equilibrium profile $\mathbf{x}^*$ is in $\mathbf{X}^H$, in which case the condition above is alone sufficient for uniqueness. However, when there is at least one strictly positive link ($g_{ij} > 0$ for some i, j), we must also consider profiles $\mathbf{x} \in \mathbf{X}^L$. The difference is that for $\mathbf{X}^L$, the BR-potential $\mathbf{P}$ need not be strictly concave in $\mathbf{x}$, even if $\alpha < 1/|\lambda_{\min}(\mathbf{G})|$. That is, the condition that $[\mathbf{I} + \alpha\mathbf{G}]$ is positive definite does not assure that $\mathbf{P}$ is strictly concave over $\mathbf{X}^L$. Nevertheless, we show directly that if there exists a Nash equilibrium in $\mathbf{X}^L$, we must have either $\alpha > 0.5$ (if the Nash equilibrium involves at least one inactive agent) or $\alpha > 0.5[(n-2)/\Delta(\mathbf{G})]$, where $\Delta(\mathbf{G}) \equiv \max_i d_i$ is the maximum node strength in the graph (if the Nash equilibrium involves all agents being active).

Before stating the result, we first introduce the definition of a best-response potential game (Voorneveld, 2000) and the BR-potential function, $\mathbf{P}$.

Definition 1. A game $\Gamma = (X_i, \pi_i)_{i=1}^n$ with strategy space $\mathbf{X} = X_1 \times \cdots \times X_n$ and payoff functions $\pi_i : \mathbf{X} \to \mathbb{R}$ for players $i \in N = \{1, \ldots, n\}$ is called a *Best-Response potential game (BR-potential game)* if there exists a function $\mathbf{P} : \mathbf{X} \to \mathbb{R}$ such that

$$\arg\max_{x_i \in X_i} \mathbf{P}(x_i, \mathbf{x}_{-i}) = \arg\max_{x_i \in X_i} \pi_i(x_i, \mathbf{x}_{-i}) \qquad [7]$$

for any $i \in N$ and any $\mathbf{x}_{-i} \in \mathbf{X}_{-i}$. The function $\mathbf{P}$ is called a *BR-potential* for Γ.

Next, we construct a BR-potential for the network contest game. Note that, for any $\mathbf{x} \in \mathbf{X}$, we let $|A(\mathbf{x})|$ denote the number of nonzero entries in the vector $\mathbf{x}$ (i.e., the number of active agents under profile $\mathbf{x}$). In addition, let $X_{tot} = \sum_h x_h$ be the sum of investments for the profile $\mathbf{x}$.

Lemma 2. *The following function,* $\mathbf{P}$*, is a BR-potential for the network contest game.*

$$\mathbf{P}(x_1, \ldots, x_n) = \begin{cases} \sum\limits_{j<k} (1 - \alpha g_{jk}) x_j x_k - \frac{1}{3}(X_{tot})^3 & \textit{if } |A(\mathbf{x})| \geqslant 2, \\ -\frac{1}{3} x_j \left[\max\limits_{i \neq j}(1 - \alpha g_{ij}) \right]^2 & \textit{if } A(\mathbf{x}) = \{j\} \\ -\frac{1}{3}\frac{n-1}{n} & \textit{if } |A(\mathbf{x})| = 0. \end{cases} \qquad [8]$$

The proof involves showing that the best responses coincide with those of the game, and closely follows the approach used by Ewerhart (2017) for the standard contest game without externalities.[14] Then, by Proposition 2.2 of Voorneveld (2000), a strategy profile $\mathbf{x}$ is a Nash equilibrium of the game if and only if it maximizes the BR-potential, $\mathbf{P}$. Therefore, if there exists a unique maximizer for $\mathbf{P}$, it is also the unique Nash equilibrium of the network contest game. We can now state our uniqueness result.

Theorem 2 (Uniqueness)**.** *Consider the network contest game with network* $\mathbf{G}$ *and externality* $\alpha \in [0, 1)$*.*

(i) *If all links are negative,* $g_{ij} \leqslant 0$ *for all* i, j*, then there is a unique Nash equilibrium if* $\alpha < 1/|\lambda_{min}(\mathbf{G})|$*; i.e., if* α *is less than the magnitude of the lowest eigenvalue of* $\mathbf{G}$*. Furthermore, when this condition holds, the unique equilibrium involves total investment* $\sum_h x_h \geqslant 0.5$*.*

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[14] Moreover, setting $\alpha = 0$ yields the same BR-potential he constructs.

(ii) *If there is at least one strictly positive link,* $g_{ij} > 0$ *for some* i, j*, then the following three conditions are when jointly satisfied, sufficient for there to exist a unique Nash equilibrium;*

(U1) $\alpha \leqslant 0.5$*;*

(U2) $\alpha \leqslant 0.5\frac{(n-2)}{\Delta(\mathbf{G})}$*; and*

(U3) $\alpha < \frac{1}{|\lambda_{min}(\mathbf{G})|}$.

Furthermore, whenever these conditions are satisfied, the unique equilibrium involves total investment $\sum_h x_h \geqslant 0.5$.

It is worth noting that, depending on the network, one of the three conditions in Theorem 2, part (ii) will always imply the other two. For instance, if $\Delta(\mathbf{G}) \equiv \max_i d_i \leqslant n-2$, then condition (U1) implies condition (U2). Otherwise, condition (U2) implies condition (U1). Similarly, if in addition to $\Delta(\mathbf{G}) \leqslant n-2$ we have $|\lambda_{min}(\mathbf{G})| \geqslant 2$, then condition (U3) is sufficient on its own. In particular then, for many networks, the condition derived by Bramoullé, Kranton and D'Amours (2014) for network games with linear best replies (our condition (U3)) is also sufficient for the network contest game.

It is also straightforward to show that these conditions are in general sufficient, but not necessary for uniqueness. Consider the complete network with $g_{ij} = 1$ for all $i \neq j$ and suppose $\alpha \in (0.5, 1)$. The lowest eigenvalue of $\mathbf{G}$ is $\lambda_{min}(\mathbf{G}) = -1$, so that condition (U3) is always satisfied. However, conditions (U1) and (U2) are (by assumption) not satisfied. Nevertheless, there exists a unique equilibrium for all values of $\alpha \in [0, 1)$; a result we establish in Section 4 as Proposition 2.

3.2 Specialized equilibria

In the next two sections of the paper, we provide results and examples characterizing the equilibria for various classes of networks, in order to highlight the interaction between properties of the network structure and the strength of the externalities in determining equilibrium investments. In several cases, we identify equilibria with a particular structure. To ease the exposition, it is convenient to define these equilibria, which we refer to as *specialized equilibria*, before introducing the different classes of networks.

Definition 2. A *specialized equilibrium* is a Nash equilibrium $\mathbf{x}^*$ in which the set of active players A forms a maximal independent set. That is, for any two players $i, j \in A$, $g_{ij} = 0$, while for every $k \in N - A$, $g_{kj} \neq 0$ for some $j \in A$.

For a given network **G** and any set of active agents A, let $d^i_A = \sum_{j \in A} g_{ij}$ denote the node strength of agent $i \in N$ *derived solely from links to active agents.* Then, define $d_{N-A,A} = \min_{i \in N-A} d^i_A$. Note that $d_{N-A,A}$ may be negative or positive. Finally, let $n_A = |A|$ denote the number of active agents in A.

Proposition 1. *Consider the game with network* **G** *and* $\alpha \in [0, 1)$.

(i) *There exists a specialized equilibrium,* $\mathbf{x}^*$*, with active agents* A *and inactive agents* $N - A$*, if and only if the subset of active players* A *is a maximal independent set,* $d_{N-A,A} > 1$*, and* $\alpha \geqslant \frac{1}{d_{N-A,A}}$.

(ii) *In any specialized equilibrium,* $x^*_i = \bar{x}_A$ *for all* $i \in A$*, where* $\bar{x}_A = \frac{n_A - 1}{n_A^2}$.

When αd^i_A is sufficiently large, an inactive player i is content to forgo competing for the prize because she can free-ride off her active neighbors and enjoy the (net) expected positive externalities that accrue if one of her neighbors wins. The greater the node strength that accrues from the active neighbors, the lower the externality can be for the inactive player to opt out of the competition, but d^i_A must always be strictly greater than one for all inactive players in order for a specialized equilibrium to exist (since $\alpha < 1$). Moreover, since $\alpha g_{ij} < 1$ for all links $(i, j) \in N \times N$, it must also be the case that every inactive agent be *positively* linked with at least two active agents.

4 Equilibria with Homogenous Links

In this section, we examine the special case of networks with *homogenous links*—network structures in which all of the (non-zero) links have identical weights. In light of the model setup, it is without loss of generality to consider just two classes of networks: (i) a *Positive* externality network, with $g_{ij} \in \{0, 1\}$ for all i, j; and (ii) a *Negative* externality network, with $g_{ij} \in \{0, -1\}$ for all i, j. Our first result characterizes the full set of equilibria for the case in which the network is *complete* (with homogenous links).

Proposition 2. *Consider the game in which* **G** *is a complete network, such that* $g_{ij} = \bar{g}$ *for all* $i \neq j$*, where either* $\bar{g} = 1$ *or* $\bar{g} = -1$*. For any* $\alpha \in [0, 1)$*, there exists a unique Nash equilibrium, in which all players are active and choose the symmetric investment level*

$$\bar{x}^\alpha = \frac{(n-1)(1-\alpha\bar{g})}{n^2}.$$

Since the proof is straightforward, we instead highlight the underlying intuition. In a complete network with homogenous links, every non-winning agent is always impacted (symmetrically) by the winning agent, rendering the externalities identity-*independent*. As a result, the game can be reformulated as a standard contest without externalities but with a modified prize value equal to the difference between the payoff from winning and the payoff from losing, which is $V - \alpha\bar{g} = 1 - \alpha\bar{g}$ (given the normalization). Uniqueness then follows from the reformulation of the game as a standard contest (for which there is always a unique equilibrium).

The comparative statics with respect to α have a natural interpretation. When the externality flows are all positive ($\alpha\bar{g} > 0$), the effective prize in the contest is reduced (by more as α increases), lowering the equilibrium investment relative to a contest without externalities. Conversely, when the externality flows are all negative ($\alpha\bar{g} < 0$), the effective prize is increased (by more as α increases), thereby increasing the equilibrium investment as players face stronger free-riding incentives. Despite this being a special case in which the network structure eliminates the identity-dependent component of the model, the basic intuition extends naturally to *symmetric* equilibria in the class of *regular networks* with homogenous links, which we discuss next.

4.1 Regular Networks with Homogenous Links

For a given network $\mathbf{G}$, we define the associated adjacency matrix $\mathbf{A_G}$ by the entries, $a_{ij} = 1$ if $g_{ij} \neq 0$ and $a_{ij} = 0$ otherwise. Thus, an adjacency matrix is *complete* if $a_{ij} = 1$ for all $i \neq j$ and $a_{ii} = 0$ for all $i \in N$. An adjacency matrix is *regular of degree* d if $\sum_j a_{ij} = d$ for all $i \in N$.

For the case of homogenous links, the adjacency matrix for a *Positive* externality network $\mathbf{G}$ with $\bar{g} = 1$ is simply $\mathbf{A} = \mathbf{G}$. Similarly, for a *Negative* externality network $\mathbf{G}$ with $\bar{g} = -1$, the associated adjacency matrix is simply $\mathbf{A} = -\mathbf{G}$. Thus, as long as links are homogenous, we refer to the network graph $\mathbf{G}$ as *regular of degree* d if $\sum_j |g_{ij}| = d$ for all $i \in N$.

The next result establishes existence of a *symmetric* equilibrium in any regular network $\mathbf{G}$ with homogenous links, for any $\alpha \in [0, 1)$.

Proposition 3. *Consider the network contest game in which the network* $\mathbf{G}$ *has homogenous links, such that* $g_{ij} \in \{0, \bar{g}\}$ *where either* $\bar{g} = 1$ *or* $\bar{g} = -1$.

Suppose $\mathbf{G}$ *is regular of degree* $d \in \{0, \ldots, n-1\}$. *Then for any* $\alpha \in [0, 1)$*, there* *exists a symmetric, pure strategy Nash equilibrium,* $\mathbf{x}^* = (x^*, \ldots, x^*)$*, where*

$$x^* \;=\; \frac{n-1-\alpha\bar{g}d}{n^2}. \qquad [9]$$

Note that, as should be expected, when $\alpha = 0$ or $d = 0$ (which is the case when $\mathbf{G}$ is the empty network), we obtain $x^* = \bar{x}$, which corresponds to the standard contest with no externalities. Furthermore, when $d = n-1$, $\mathbf{G}$ is the complete network and we obtain $x^* = \bar{x}^{\alpha}$.

In addition, comparative statics with respect to α and d have natural and intuitive interpretations. For positive externalities ($\bar{g} = 1$), free-riding incentives reduce the equilibrium investment compared to a standard contest without externalities. For negative externalities ($\bar{g} = -1$), the effective value of winning the contest increases so that competition intensifies, pushing equilibrium investment higher than in the standard contest. For both positive and negative externalities, these effects are amplified as α (the strength of the externality) increases, and as d increases, which corresponds to an increase in network density (from the empty network when $d = 0$, to the complete network when $d = n-1$).

When externalities are negative ($\bar{g} = -1$), the symmetric equilibrium is also unique (by Theorem 2 and $\alpha < 1$). However, unlike the complete network, for positive externalities, the symmetric equilibrium in Proposition 3 may not be the only equilibrium. Combining Proposition 1 with Proposition 3, it follows that for *regular networks* with homogenous links, there may exist both a *symmetric* equilibrium and a *specialized* equilibrium. More concretely, whenever the graph has a maximal independent set A with $\alpha \geqslant 1/d_{N-A,A}$, there exists a specialized equilibrium.[15] Moreover, in many cases, there may exist multiple specialized equilibria corresponding to different maximal independent sets of agents. To illustrate this multiplicity, we present two examples of regular networks with positive externalities, and highlight the ranges of α for which there exist both specialized equilibria and a symmetric equilibrium with $A = N$.

Example 1. [A circle network with $n = 6$] In the circle network, the players are arranged around a circle and linked to the two agents on either side. Suppose the externalities are positive, such that $\bar{g} = 1$. Thus, the circle network is regular of degree $d = 2$. By Proposition 3, there exists a symmetric equilibrium for any

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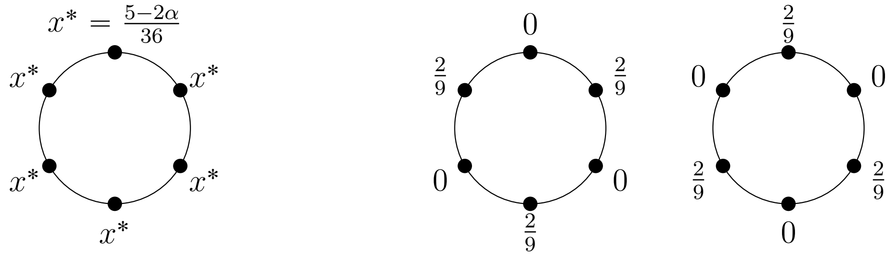
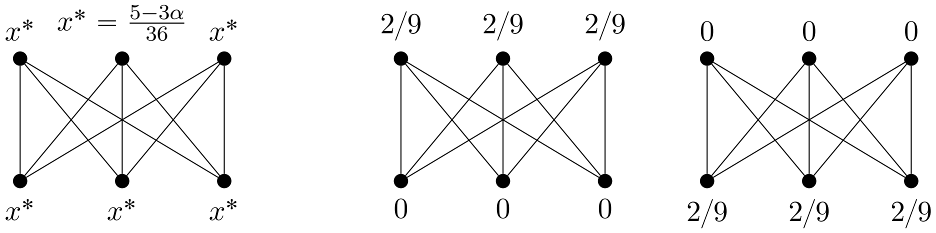
[15] Note that in some cases, such a maximal independent set may not exist. For instance, consider the circle network with $n = 5$ agents. In this network, every maximal independent set is of order at most one, meaning that there is always at least one inactive agent who is connected to only one active agent, i.e., $d_{N-A,A} \leqslant 1$. In this case, a specialized equilibrium does not exist for any $\alpha < 1$.



(a) Symmetric equilibria, $\alpha \in [0, 1)$ **(b)** Specialized equilibria, $\alpha \in [0.5, 1)$

Figure 1. Equilibria in the circle network with $n = 6$ agents. Panel (a): A symmetric equilibrium with all agents active exists for any $\alpha \in [0, 1)$. Panel (b): When $\alpha \geqslant 0.5$, there are two specialized equilibria, each characterized by a maximal independent set of three agents, with each active agent investing $\bar{x}_A = 2/9$.



(a) Symmetric equilibria, $\alpha \in [0, 1)$ **(b)** Specialized equilibria, $\alpha \in [1/3, 1)$

Figure 2. Equilibria in the complete bipartite network with $n = 6$ agents. Panel (a): A symmetric equilibrium with all agents active exists for any $\alpha \in [0, 1)$. Panel (b): When $\alpha \geqslant 1/3$, there are two specialized equilibria, each characterized by a maximal independent set of three agents, with each active agent investing $\bar{x}_A = 2/9$.

$\alpha \in [0, 1)$, in which all agents are active and each invests $x^* = \frac{5-2\alpha}{36}$; see panel (a) in Figure 1. Moreover, for $n = 6$, there are two maximal independent sets, as shown in Figure 1, panel (b). For each of these, $n_A = 3$, so that each active agent invests $\bar{x}_A = 2/9$. Furthermore, since every inactive player is linked to two active players, $d_{N-A,A} = 2$. Thus, the specialized equilibria exist if and only if $\alpha \geqslant 0.5$.

Example 2. [A bipartite network with $n = 6$] $\mathbf{G}$ is a bipartite graph if the nodes (agents) can be partitioned into two disjoint sets A and B, with $g_{ij} = 0$ for all $i, j \in A$ and $g_{kl} = 0$ for all $k, l \in B$. Figure 2 illustrates a complete bipartite graph with $n = 6$ agents. Suppose externalities are positive, such that $\bar{g} = 1$. Then the network is regular of degree $d = 3$. By Proposition 3, there exists a symmetric equilibrium for any $\alpha \in [0, 1)$, in which all agents are active and each invests $x^* = \frac{5-3\alpha}{36}$; see panel (a) in Figure 2. Moreover, the three agents on the

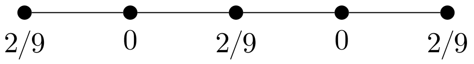


Figure 3. A specialized equilibrium for the line network with $n = 5$ agents exists if and only if $\alpha \geqslant 0.5$. The center agent and the agents at the endpoints of the line form a maximal independent set. Each active agent invests $\bar{x}_A = 2/9$.

top and the three agents on the bottom represent the two maximal independent sets (as well as the two elements of the partition); see panel (b) in Figure 2. Given $n_A = 3$, each active agent invests $\bar{x}_A = 2/9$. Since the graph is a complete bipartite graph, each inactive agent in a specialized profile is linked to all of the active agents, so that $d_{N-A,A} = 3$. Thus, the specialized equilibria shown exist if and only if $\alpha \geqslant 1/3$.

Although the prior examples illustrate specialized equilibria in the context of regular networks, specialized equilibria may also arise in other classes of networks. To underscore this point, consider the example of a line network with $n = 5$ agents and homogenous links, which is not regular.[16]

Example 3 (A line network). Suppose externalities are positive ($\bar{g} = 1$). In the line network, whenever n is odd, there is a specialized equilibrium associated with the maximal independent set consisting of the endpoints of the line and every second node in between (see Figure 3). Every inactive agent is connected to two active agents, so that $d_{N-A,A} = 2$. Thus, the specialized equilibrium exists if and only if $\alpha \geqslant 0.5$.

Further examples of specialized equilibria arise in the context of another commonly studied class of networks; those that exhibit a core-periphery structure.

4.2 Core-Periphery Networks with Homogenous Links

The class of *core-periphery networks* is comprised of networks consisting of two types of agents—a set of highly connected *core* players, and a set of less connected *periphery* players. While this class of networks is very broadly defined, we restrict attention to a subset of the class that includes many of the most commonly studied core-periphery structures.

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[16] Note that, for a line network with an even number of agents (n even), if the set of active agents forms a maximal independent set, there is always at least one inactive agent who is linked to just one active agent. Thus, by Proposition 1, there does not exist a specialized equilibrium for the line if n is even.

In particular, we define a subclass of core-periphery structures referred to as *core-to-periphery* networks.

Definition 3. In a *core-to-periphery* network,

(i) there are $n_c \geqslant 1$ core players,

(ii) all core players are connected to each other, creating a dense, or completely connected core,

(iii) each core player is connected to $m \geqslant 1$ periphery players,

(iv) and each periphery player is connected to a *single* core player and no other periphery players.

Thus, there are $n = n_c(1+m)$ total players, comprised of $n_c m$ periphery players, all with degree $d_p = 1$, and n_c core players, each with degree $d_c = (n_c - 1) + m$.

The conditions laid out in Definition 3 are satisfied by, for instance, the *star network*, which has a single core player ($n_c = 1$) connected to m periphery players. For all such *core-to-periphery* networks, we characterize the semi-symmetric equilibrium in which all players of the same type choose identical levels of investment. We denote the investment levels by x_c and x_p for core and periphery players, respectively.

Proposition 4. *Consider the game defined by* $\alpha \in [0,1)$ *and the network* $\mathbf{G}$*, for which links are homogenous, such that* $g_{ij} \in \{0, \bar{g}\}$ *(where* $\bar{g}$ *is either* 1 *or* -1*).*

Suppose $\mathbf{G}$ *is a core-to-periphery network with* n_c *core players, each connected to* m *peripheral players. Then there exists a semi-symmetric, pure strategy Nash equilibrium in which every core player chooses the same investment* x_c^**, and every peripheral player chooses the same investment* x_p^**, where*

(i) *if* $\alpha\bar{g} < \frac{1}{m}$*, then* $x_c^* = [1 - \alpha\bar{g}m]\Delta$ *and* $x_p^* = [1 + \alpha\bar{g}(n_c - 2)]\Delta$ *, where*

$$\Delta = \frac{n_c[1 + m + \alpha\bar{g}m(n_c - 3)] - [1 + \alpha\bar{g}(n_c - 1 - \alpha\bar{g}m)]}{n_c^2[1 + m + \alpha\bar{g}m(n_c - 3)]^2} \geqslant 0.$$

(ii) *if* $\alpha\bar{g} \geqslant \frac{1}{m}$*, then* $x_c^* = 0$ *and* $x_p^* = \frac{n_c m - 1}{(n_c m)^2}$.

Note that when $\alpha = 0$, the equilibrium investments reduce to the standard contest equilibrium,

$$x_c^* = x_p^* = \frac{n_c(1+m) - 1}{n_c^2(1+m)^2} = \frac{n-1}{n^2}.$$

For negative externalities and sufficiently small, positive externalities ($\alpha\bar{g} < 1/m$), the semi-symmetric equilibrium is interior; that is, both sets of agents are active. In addition, the semi-symmetric equilibrium investment for core players is decreasing in the externality (and strictly decreasing until they become inactive). In contrast, for periphery players, equilibrium investment is non-monotonic in $\alpha\bar{g}$.

Moreover, for $\alpha\bar{g} < 0$, we have $x_c^* > x_p^*$. Intuitively, the core players are structurally more *exposed* to the negative externality than are the less connected periphery players (who are linked only to a single core agent, by assumption). Accordingly, for $\alpha\bar{g} > 0$, free-riding incentives are also stronger for core players than for periphery players, so that $x_c^* < x_p^*$ in the semi-symmetric equilibrium with positive externalities.

When the positive externality becomes sufficiently large ($\alpha\bar{g} \geqslant 1/m$), the semi-symmetric equilibrium is a *specialized equilibrium*. Free-riding incentives for the core players are sufficiently strong that they choose to be inactive in the contest. When this is the case, only the periphery players are active, and since they are not connected to each other, they form a maximal independent set and their equilibrium investment coincides with the equilibrium for a standard contest between $n_c m$ players (i.e., the total number of periphery players). Thus, for the subclass of core-to-periphery network structures, strong positive externalities lead to polarization of competition in the semi-symmetric equilibrium.

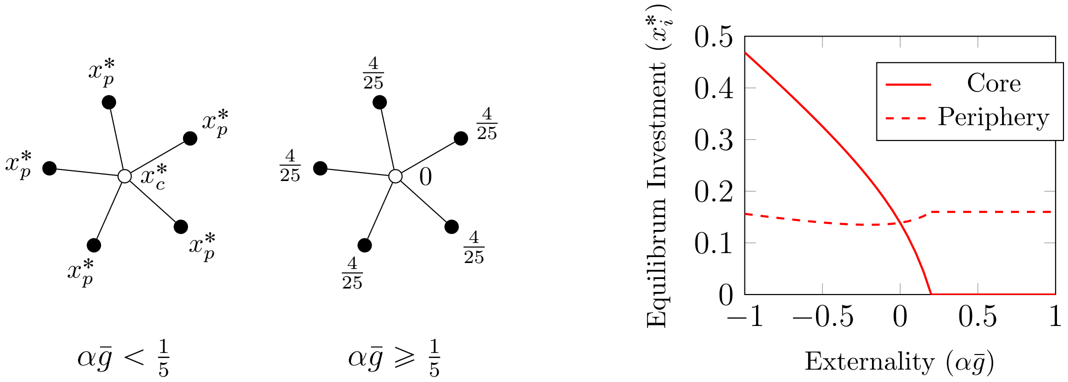
The following examples serve to illustrate the semi-symmetric equilibria in two common core-to-periphery network structures.

Example 4 (A star network). In a star network, there is a single core player, such that $n_c = 1$, and m peripheral players connected to the core (see Figure 4a where the core player is distinguished by the hollow node). For $m = 5$, the semi-symmetric equilibrium involves full participation when $\alpha\bar{g} < \frac{1}{5}$, with

$$x_c^* = \frac{5(1-5\alpha\bar{g})(1-\alpha\bar{g})^2}{4(3-5\alpha\bar{g})^2} \qquad \text{and} \qquad x_p^* = \frac{5(1-\alpha\bar{g})^3}{4(3-5\alpha\bar{g})^2}.$$

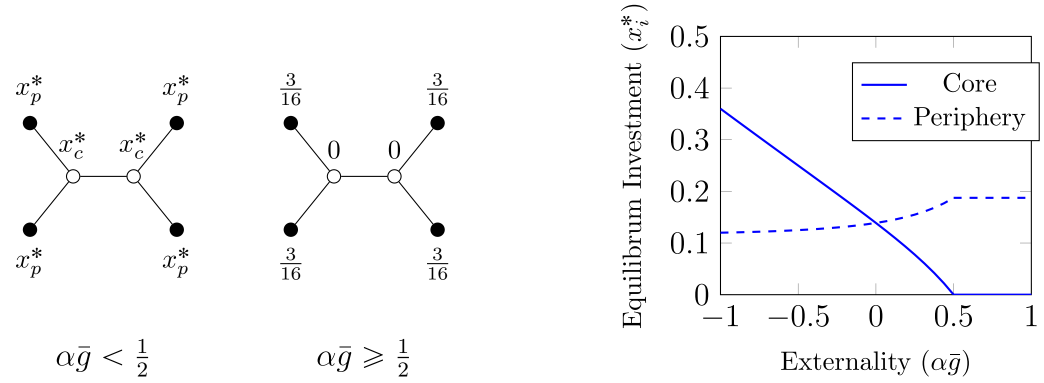
When $\alpha\bar{g} \geqslant \frac{1}{5}$, the semi-symmetric equilibrium is a specialized equilibrium with A equal to the set of peripheral players, with $x_c^* = 0$ and $x_p^* = \frac{4}{25}$. Figure 4 shows the two cases on the network grap,h in panel (a), and in a graph that plots the equilibrium investment against $\alpha\bar{g}$ for both player types, in panel (b).

Example 5. [A core-periphery network with $n_c = 2$] In the CP2 network (see Figure 5a), there are $n_c = 2$ core players (distinguished by hollow nodes), each



(a) Star network, for $\alpha\bar{g} < \frac{1}{5}$ and $\alpha\bar{g} \geqslant \frac{1}{5}$ **(b)** Equilibrium investment, $\alpha\bar{g} \in (-1, 1)$

Figure 4. Semi-symmetric equilibria in the star network.



(a) CP2 network, for $\alpha\bar{g} < \frac{1}{2}$ and $\alpha\bar{g} \geqslant \frac{1}{2}$ **(b)** Equilibrium investment, $\alpha\bar{g} \in [-1, 1)$

Figure 5. Semi-symmetric equilibria in the CP2 network.

connected to 2 peripheral players. Thus, the semi-symmetric equilibrium involves full participation when $\alpha\bar{g} < \frac{1}{2}$, with

$$x_c^* = \frac{(1 - 2\alpha\bar{g})(5(1 - \alpha\bar{g}) + 2\alpha^2)}{4(3 - 2\alpha\bar{g})^2} \qquad \text{and} \qquad x_p^* = \frac{5(1 - \alpha\bar{g}) + 2\alpha^2}{4(3 - 2\alpha\bar{g})^2},$$

and is the specialized equilibrium with $x_c^* = 0$ and $x_p^* = \frac{3}{16}$ whenever $\alpha\bar{g} \geqslant \frac{1}{2}$. These equilibria are again illustrated on the network graph and plotted against $\alpha\bar{g}$ in panels (a) and (b) of Figure 5.

5 Equilibria with Homogenous Node Strengths

In this section, we relax the assumption that links are homogenous. Nevertheless, we retain some homogeneity by restricting attention to networks in which each player has the same total *node strength*, $d_i = \sum_j g_{ij} = k$. While it is more difficult to establish general properties of equilibria with the additional heterogeneity, we introduce several illustrations to highlight equilibrium characteristics in this more general setting.

5.1 Complete adjacency networks

We focus initially on networks $\mathbf{G}$ for which the adjacency matrix $\mathbf{A_G}$ is complete. That is, every possible link is non-zero (excluding self-loops); formally, for every i, j with $i \neq j$, we have $g_{ij} \neq 0$.

The following example demonstrates that the introduction of link heterogeneity can result in multiple equilibria even when the underlying adjacency matrix is complete and the links balance out perfectly, such that every agent's node strength is $d_i = 0$.

Example 6. Suppose $n = 6$ and that for each agent i, there are two other players for whom $g_{ij} = 1/3$, two other players for whom $g_{ij} = 1/6$, and one player (the *rival*) for whom $g_{ij} = -1$. The network is depicted in Figure 6. It follows that $d_i = 0$ for all $i \in N$. In the resulting network contest game, there is a symmetric equilibrium in which all players choose the same equilibrium investment as in a standard contest, $\bar{x} = (n-1)/n^2$.

However, this need not be the unique equilibrium. For the given network $\mathbf{G}$, the lowest eigenvalue is $\lambda_{min}(\mathbf{G}) = -\sqrt{5}$. Notice then that for any $\alpha \geqslant 1/|\lambda_{min}(\mathbf{G})| = 1/\sqrt{5}$, the sufficient condition (U3) of Theorem 2 is violated. In particular, the following constitutes an equilibrium, provided $\alpha \geqslant 2/3$: let $A = \{1, 4\}$ be the set of active players, noting that $g_{1,4} = -1$. That is, the two active players are mutual rivals. Each inactive player is linked to the two active players—one with positive weight 1/3 and the other with positive weight 1/6.

By Lemma 1, the equilibrium investment for the two active players is $x_i^* = (1+\alpha)/4$, and thus, the inactive players, $N - A$, will choose to invest zero so long as

$$\Big(2 - \alpha(1/3 + 1/6)\Big)\frac{1+\alpha}{4} \leqslant \left(\frac{1+\alpha}{2}\right)^2$$

which reduces to $\alpha \geqslant 2/3$. Similar equilibria can be constructed in which the two active players are $A = \{2, 5\}$ or $A = \{3, 6\}$.

Thus, in addition to the symmetric equilibrium in which all agents are active, there exists an asymmetric equilibrium for each pair of *mutual rivals*, in which competition is entirely localized to the selected pair.

Building on the intuition provided by Example 6, similar characteristics of

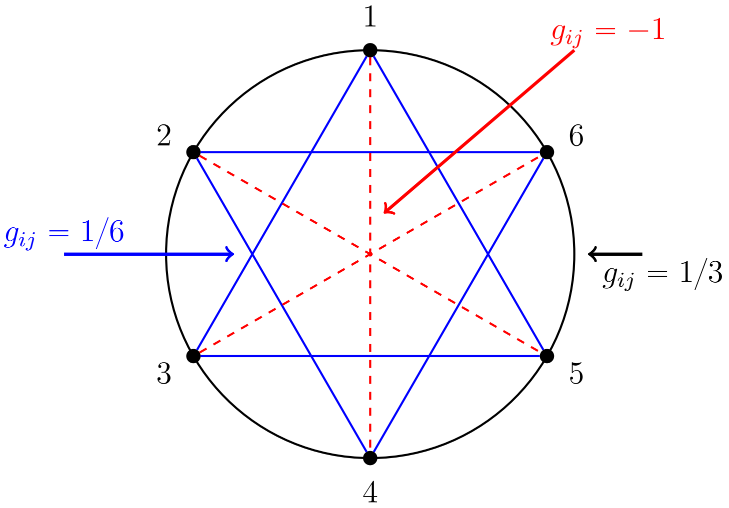


Figure 6. Complete Adjacency matrix, with Homogenous node strength, $d_i = 0$, for all $i \in N$. Link weights around the outer ring are $g_{ij} = 1/3$, solid (blue) interior link weights are $g_{ij} = 1/6$, while the dashed (red) interior link weights are $g_{ij} = -1$.

equilibria emerge for homogenous *non-zero* node strengths. Suppose $d_i = k \neq 0$ for all $i \in N$. It is straightforward to show that there always exists a symmetric equilibrium, in which each player invests

$$\bar{x}^k = \frac{(n-1)(1-\alpha k)}{n^2}.$$

For α sufficiently small, Theorem 2 guarantees that this will be the unique equilibrium. However, for sufficiently large α, there may exist asymmetric equilibria in which some players are inactive.

Example 7. Consider the example network depicted in Panel (a) of Figure 7. Each player has one "enemy" (with $g_{ij} = -0.5$), one strong "ally" (with $g_{ij} = +1$), and three moderate allies (with $g_{ij} = +0.5$). Thus, $d_i = +2$ for all agents. In addition to the fully symmetric equilibrium, there is an equilibrium in which two rivals compete, with all other players remaining inactive, provided α is sufficiently large. For example, let $A = \{1, 2\}$ and note that, by Lemma 1, $x_1 = x_2 = (1 + 0.5\alpha)/4$ and $x_j = 0$ for $j \in \{3, 4, 5, 6\}$ is an equilibrium investment as long as $\alpha \geqslant 2/3$.[17]

The network shown in Panel (b) of Figure 7, is the "negative" of the one shown in Panel (a), with each player connected to one "ally" (with $g_{ij} = +0.5$), one strong "enemy" (with $g_{ij} = -1$), and three moderate enemies (with $g_{ij} = -0.5$). Thus,

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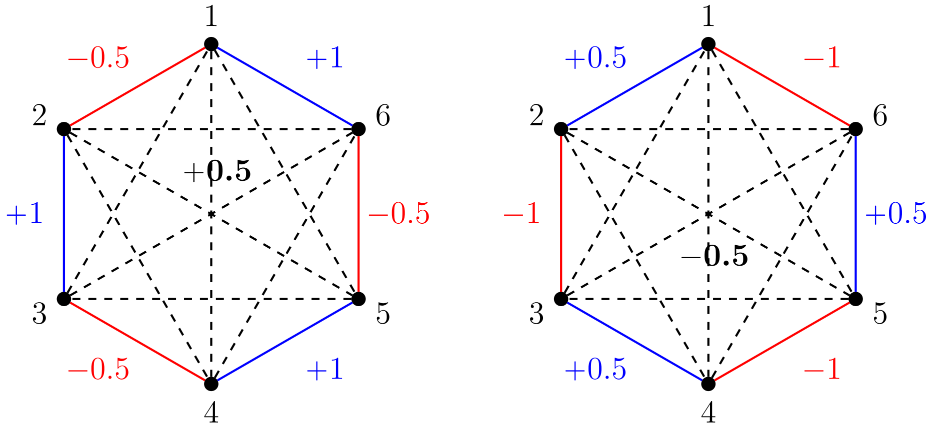
[17] In fact, there is one such equilibrium for each pair of "enemies" in the network.



(a) Positive Node Strengths, $k = +2$. **(b)** Negative Node Strengths, $k = -2$.

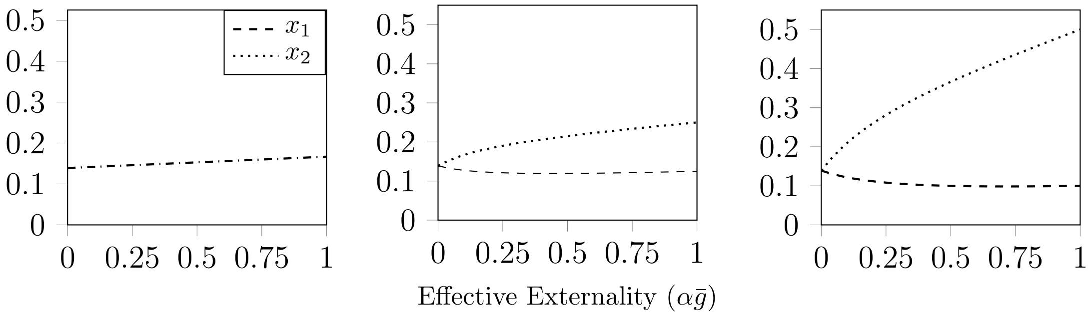
Figure 7. Complete Adjacency matrix, with Homogenous node strength, $d_i = k$, for all $i \in N$. Panel (a): $k = 2$. Link weights around the outer ring alternate between $g_{ij} = -0.5$ and $g_{ij} = +1$, while all interior link weights are $g_{ij} = +0.5$. Panel (b): $k = -2$. Link weights around the outer ring alternate between $g_{ij} = +0.5$ and $g_{ij} = -1$, while all interior link weights are $g_{ij} = -0.5$.

$d_i = -2$ for all players. Unlike for Panel (a), for the network in Panel (b) of Figure 7, the fully symmetric equilibrium is the unique Nash equilibrium of the resulting network contest game.

5.1.1 Application—A Model of Competing Alliances

Here, we consider an "alliance and enmities" environment in which the set of players, N, is partitioned into two disjoint sets, N_1 and N_2, consisting of n_1 and n_2 individuals, respectively. The network $\mathbf{G}$ is such that $g_{ij} = \bar{g} > 0$ if $(i, j) \in N_1 \times N_1$ or $(i, j) \in N_2 \times N_2$; with $g_{ij} = -\bar{g}$ otherwise. In this application, with two factions of players, a positive link weight indicates that two individuals are allies while a negative link weight indicates they are enemies. We posit the existence of a fully interior, semi-symmetric equilibrium in which each individual exerts the same effort level as each of their allies (those in the same subset of N), but where effort levels may differ between enemies (i.e., between the two factions).

Let x_1 and x_2 denote the effort level for individuals in N_1 and N_2, respectively. Then, the equilibrium conditions described in Lemma 1 yield the following relationship: $[1 + \alpha\bar{g}(2n_1 - 1)]\, x_1 = [1 + \alpha\bar{g}(2n_2 - 1)]\, x_2$. Notice that if $n_1 = n_2$ (i.e., alliances of equal size), the preceding condition implies that $x_1 = x_2$ and the equilibrium is fully symmetric. Conversely, when the number of members in competing alliances differs, the equilibrium investment levels of individuals in each alliance



(a) $n_1 = 3, n_2 = 3$ **(b)** $n_1 = 4, n_2 = 2$ **(c)** $n_1 = 5, n_2 = 1$

Figure 8. Equilibrium investment levels for individuals in competing alliances.

no longer coincide. In general, we obtain the following equilibrium predictions:

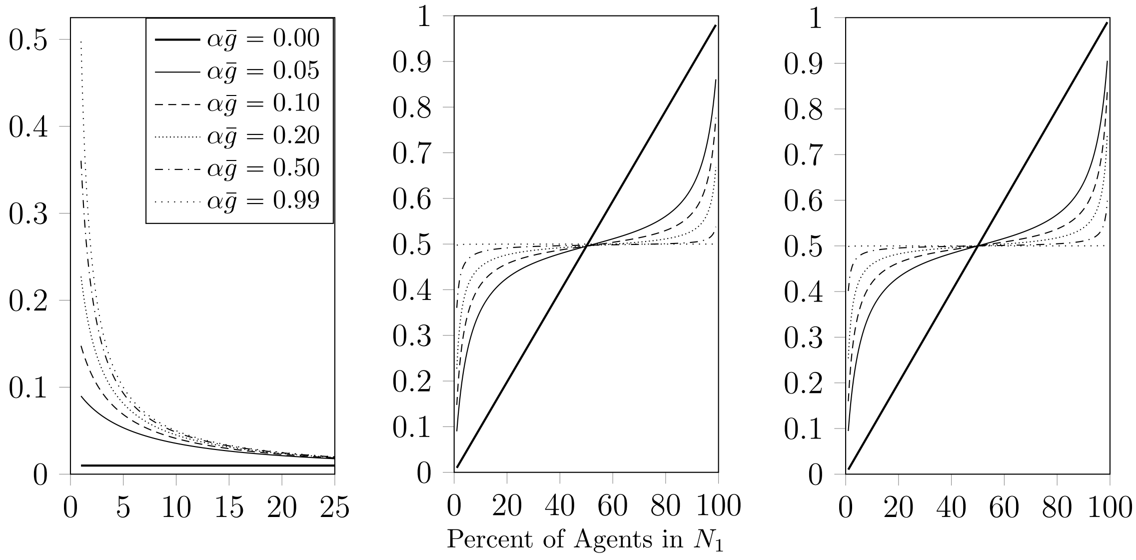
$$x_1 = [1 + \alpha\bar{g}(2n_2 - 1)] \left[\frac{(n_1 + n_2 - 1)(1 - \alpha\bar{g})^2 + 4n_1 n_2 \alpha\bar{g}}{[(n_1 + n_2)(1 - \alpha\bar{g}) + 4n_1 n_2 \alpha\bar{g}]^2} \right]$$
$$x_2 = [1 + \alpha\bar{g}(2n_1 - 1)] \left[\frac{(n_1 + n_2 - 1)(1 - \alpha\bar{g})^2 + 4n_1 n_2 \alpha\bar{g}}{[(n_1 + n_2)(1 - \alpha\bar{g}) + 4n_1 n_2 \alpha\bar{g}]^2} \right]$$

EXAMPLE 8. Suppose $n = 6$ and each pair of agents are either allies with $g_{ij} = \bar{g} > 0$ or enemies with $g_{ij} = -\bar{g}$. The equilibrium investment levels of individuals in each subset of N depend on the fraction of agents that are in their alliance. The graphs in Figure 8 illustrate the equilibrium investment level of an individual in each alliance for various combinations of alliance sizes. The effective externality ($\alpha\bar{g}$) is restricted to $[0, 1)$ since negative externalities are captured by the fact that $g_{ij} = -\bar{g}$ for enemies. As the sizes of the two alliances diverge, so too do the equilibrium investments for the members of each alliance, with the difference increasing in the effective externality, $\alpha\bar{g}$. Intuitively, this divergence is driven primarily by a sharper increase in equilibrium investment by agents in the smaller alliance, whereas the agents in the larger alliance are less sensitive to the effective externality.

To further examine the impact of differing alliance sizes on equilibrium investment levels, suppose we normalize $n = 100$ and think of n_1 as denoting the percentage of agents in N_1. Then, we can express the equilibrium effort level of any agent $i \in N_1$ as follows:

$$x_1 = [1 + \alpha\bar{g}(199 - 2n_1)] \left[\frac{99(1 - \alpha\bar{g})^2 + 4n_1(100 - n_1)\alpha\bar{g}}{[100(1 - \alpha\bar{g}) + 4n_1(100 - n_1)\alpha\bar{g}]^2} \right]$$

The graphs in Figure 9 illustrate, for various effective externality values, how the



(a) x_1 **(b)** $n_1 x_1$ **(c)** $n_1 P(x_1, \mathbf{x})$

Figure 9. The impact of alliance size on equilibrium investment, aggregate equilibrium investment within-alliance, and probability of alliance member winning the contest.

individual investment, aggregate investment by all members of the alliance N_1, and the probability that a member of the alliance N_1 wins the contest change with respect to the percentage of agents in the alliance.

As the effective externality grows in magnitude—that is, the strength of alliances and enmities alike increases—population imbalances become less pivotal for the outcome of the contest. For instance, when the effective externality is $\alpha\bar{g} = 0.5$, the equilibrium probability of some player from N_1 winning the contest is very close to 0.5 whether the N_1 alliance includes 10% of the population of agents, or 90% of the population of agents. Only more extreme imbalances between the groups can generate a clear advantage for one group over another. In contrast, when the effective externality is relatively weak, asymmetry in the size of the groups has a greater impact on the equilibrium probability with which the winning agent comes from each group.

5.1.2 Relation to Group Contests

Our model can also be adapted, in the limit, to represent a standard model of group contests. To see this, consider a model in which there are n agents assigned to m different groups, $N_1, \ldots, N_m$. Within each group N_k, let $g_{ij} = \bar{g} = 1$ for all $i, j \in N_k$. Meanwhile, suppose that if i and j are from different groups, $g_{ij} =$

0.[18] Assume that the prize is a group-specific public good. That is, consider the limit case of our model where $\alpha = 1$, so that each group member's payoff is equal to $V = 1$ in the event that any one of the group's members wins the contest.[19] Assuming a lottery contest success function is applied to aggregate group performance, measured as the sum of group members' investments (i.e., where individual group members' efforts are perfect substitutes), the equilibrium *group-level* investment is unique. However, given that all group members have the same valuation for the public good prize, there are multiple configurations of individual investment levels that generate this aggregate group-level investment (see, e.g. Baik, 1993, 2008).

5.2 Incomplete Adjacency Networks

We now consider networks $\mathbf{G}$ for which the adjacency matrix $\mathbf{A}_{\mathbf{G}}$ is incomplete. That is, networks in which there are some pairs of agents i, j with $i \neq j$ such that $g_{ij} = 0$. Specifically, we consider a circle network and complete bipartite network where link weights can take one of two values, w_1 and w_2. Node strengths are $d_i = w_1 + w_2$ for all $i \in N$ in the circle network and $d_i = 2w_1 + w_2$ in the bipartite network. Figure 10 shows the graphs of these networks.

On the circle network, the network contest game has a symmetric equilibrium in which all players are active and invest $x^* = \frac{5-\alpha(w_1+w_2)}{36}$. This equilibrium exists for any $\alpha \in [0, 1)$ since $5 > \alpha(w_1 + w_2)$ for any $w_1, w_2 \in [-1, 1]$. Additionally, there are two specialized equilibria which exist for suitable combinations of α, w_1, and w_2. The network has two maximal independent sets $M_1 = \{1, 3, 5\}$ and $M_2 = \{2, 4, 6\}$. In a specialized equilibrium, all agents in one maximal independent set are active and invest $\bar{x}_A = 2/9$, while all agents in the other set are inactive. Since each inactive agents is connected to two active players, who invest the same amount, these specialized equilibria exist if and only if $\alpha d_i = \alpha(w_1 + w_2) \geqslant 1$.

The network contest game on the complete bipartite network also has both an interior, symmetric equilibrium, and two specialized equilibria. In the symmetric equilibrium, which exists for any α, each player invests $x^* = \frac{5-2\alpha w_1-\alpha w_2}{36}$. As in the circle network, the agents who are active in a specialized equilibrium on the complete bipartite network are all members of the same maximal independent set, $M_1 = \{1, 2, 3\}$ or $M_2 = \{4, 5, 6\}$. In a specialized equilibrium, each active agent invests $\bar{x}_A = 2/9$. Since each inactive agent is connected to three active players,

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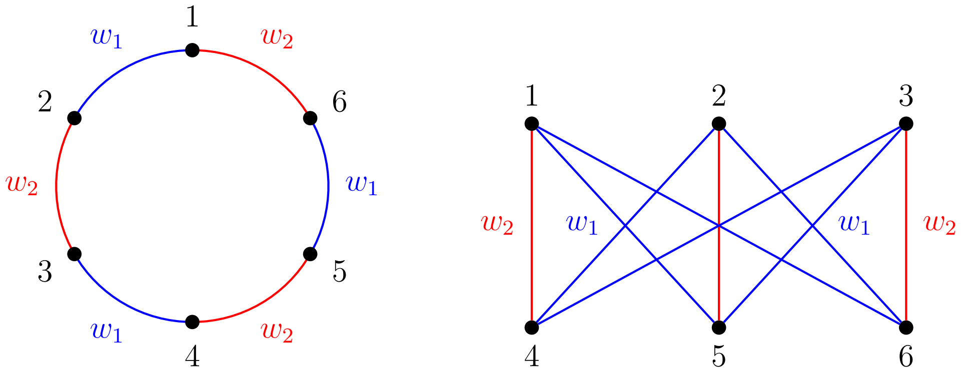
[18] Strictly speaking, this means that the adjacency network associated with $\mathbf{G}$ is incomplete.

[19] Recall, our general results are only stated for $\alpha \in [0, 1)$ and so need not extend directly to the case where $\alpha = 1$.



(a) Circle Network **(b)** Bipartite Network

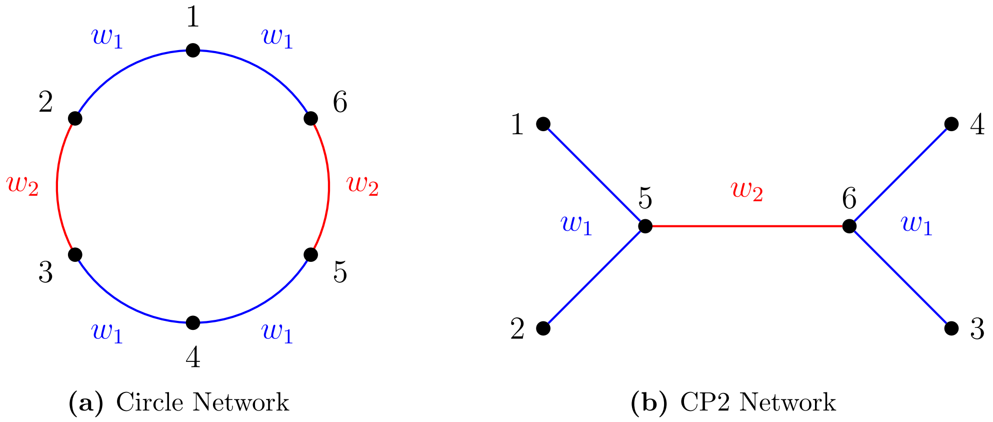
Figure 10. Networks with homogeneous node strength and heterogeneous link weights. Link weights are either $g_{ij} = w_1$ (blue) or $g_{ij} = w_2$ (red). Panel (a): Circle Network with node strengths of $d_i = w_1 + w_2$ for all $i \in N$. Panel (b): Bipartite network with node strengths of $d_i = 2w_1 + w_2$, for all $i \in N$.

who all invest the same amount, the specialized equilibria exist if and only if $\alpha d_i = \alpha(2w_1 + w_2) \geqslant 1$.

Overall, these results are not too dissimilar from those presented earlier for the fully homogeneous circle network (Example 1) and fully homogeneous bipartite network (Example 2). However, introducing link weight heterogeneity, while maintaining node strength homogeneity, appears to have two main effects. First, the predictions for symmetric equilibrium investment levels now depend separately on the weight of each link connecting an individual to one of their neighbors. A change in either link weight will shift the predicted investment level by an amount proportional to the change in weight of the link. Second, while the threshold conditions for existence of a specialized equilibrium still depend on the node strength of inactive agents, their node strength now depends separately on the weights of the links connecting them to active agents. Changes in the weight of any link between an active and inactive agent will affect the threshold value of α for which specialized equilibria exist.

6 Equilibria with Heterogeneous Links and Node Strengths

While our general results presented in Section 3 hold for any network structure, deriving closed-form solutions and general properties for network contest games on fully heterogeneous networks is not straightforward due to the inherent complexity of such an environment. To provide some intuition regarding how predictions change in this most general environment, this section examines a selection of networks with $n = 6$ agents in which both links and node strengths are heterogeneous.



(a) Circle Network **(b)** CP2 Network

Figure 11. Networks with heterogeneous link weights and node strengths. Link weights are either $g_{ij} = w_1$ (blue) or $g_{ij} = w_2$ (red). Panel (a): Circle Network with node strengths of $d_1 = d_4 = 2w_1$ and $d_2 = d_3 = d_5 = d_6 = w_1 + w_2$. Panel (b): CP2 network with node strengths of $d_1 = d_2 = d_3 = d_4 = w_1$ and $d_5 = d_6 = 2w_1 + w_2$.

For tractability and simplicity, we assume that link weights take one of two values, w_1 or w_2, in all networks.

We examine two networks which are heterogeneous versions of network structures considered in previous sections. First, we consider a circle network with heterogeneous node strengths of $d_1 = d_4 = 2w_1$ and $d_2 = d_3 = d_5 = d_6 = w_1 + w_2$. Panel (a) of Figure 11 depicts the graph of this network. Note that, while the basic structure remains the same, this version of the circle network no longer satisfies our definition of regular networks since $d_i = \sum_j |g_{ij}|$ is not the same for all $i \in N$. The second network is a heterogeneous version of the CP2 network, which was introduced in Example 5, with node strengths of $d_1 = d_2 = d_3 = d_4 = w_1$ and $d_5 = d_6 = 2w_1 + w_2$. Panel (b) of Figure 11 shows the graph of this network.

In contrast with other versions of the circle network previously considered, which maintain node strength homogeneity, the interior equilibrium of the network contest game on this heterogeneous circle network is not symmetric. Rather, the interior equilibrium is semi-symmetric with individuals of the same node strength investing at the same level. The semi-symmetric equilibrium investment levels for each agent in this network are described below. Note that it is necessarily the case that $1 - \alpha w_1 > 0$, and therefore, this equilibrium exists if and only if $1 - 2\alpha w_1 + \alpha w_2 > 0$ and $2\alpha^2 w_1^2 - 8\alpha w_1 + \alpha w_2 + 5 > 0$ hold.

$$\begin{aligned} x_1^* = x_4^* &= (1 - 2\alpha w_1 + \alpha w_2)\left[\frac{2\alpha^2 w_1^2 - 8\alpha w_1 + \alpha w_2 + 5}{4(3 - 4\alpha w_1 + \alpha w_2)^2}\right] \\ x_2^* = x_3^* = x_5^* = x_6^* &= (1 - \alpha w_1)\left[\frac{2\alpha^2 w_1^2 - 8\alpha w_1 + \alpha w_2 + 5}{4(3 - 4\alpha w_1 + \alpha w_2)^2}\right] \end{aligned} \qquad [10]$$

The heterogeneous circle network also has two specialized equilibria in which the set of active agents corresponds to one of the two maximal independent sets of agents, $M_1 = \{1, 3, 5\}$ and $M_2 = \{2, 4, 6\}$. In a specialized equilibrium, all agents in one maximal independent invest $\bar{x}_A = 2/9$, while those in the other set are inactive. Since node weights are heterogeneous even within a maximal independent set, there are now two conditions that must be satisfied for a specialized equilibrium to exist. In the specialized profile where $A = M_1$, the equilibrium conditions for inactive agents are $\alpha w_1 + \alpha w_2 \geqslant 1$ (agents 2 and 6) and $2\alpha w_1 \geqslant 1$ (agent 4). The specialized equilibrium exists if and only if both conditions are satisfied; that is, if and only if $\min\{\alpha w_1 + \alpha w_2, 2\alpha w_2\} \geqslant 1$. This same condition is required for existence of the other specialized equilibrium, where $A = M_2$, since $d_2 = d_3 = d_5 = d_6$ and $d_1 = d_4$.

In the heterogeneous CP2 network, as in the homogeneous version of this network, the fully interior equilibrium is semi-symmetric with equilibrium investment levels being the same within type (core or periphery) and differing across types. The semi-symmetric equilibrium for the heterogeneous CP2 network is described below.

$$\begin{aligned} x_1^* = x_2^* = x_3^* = x_4^* &= (1 - 2\alpha w_1)\left[\frac{2\alpha^2 w_1^2 - 8\alpha w_1 + 3\alpha w_2 + 5}{4(4\alpha w_1 - 2\alpha w_2 - 3)^2}\right] \\ x_5^* = x_6^* &= [1 - \alpha(w_1 - w_2)]\left[\frac{2\alpha^2 w_1^2 - 8\alpha w_1 + 3\alpha w_2 + 5}{4(4\alpha w_1 - 2\alpha w_2 - 3)^2}\right] \end{aligned} \qquad [11]$$

There is also a specialized equilibrium in which only peripheral players are active and each invests $\bar{x}_A = 3/16$, while the two core players drop out of the contest. This equilibrium exists if and only if $2\alpha w_1 \geqslant 1$. That is, the existence of the specialized equilibrium requires that the effective externality obtained by core players, from their connections with peripheral players, is sufficiently high.

These results suggest two main implications for the structure of equilibria in fully heterogeneous networks. First, the symmetry of interior equilibria is contingent on node strength symmetry. When node strengths are heterogeneous, interior equilibria become semi-symmetric with investment levels depending on an individual's node strength. This is true even when we maintain regularity in the sense that each individual has the same number of neighbors, as in the heterogeneous circle network considered in this section. The CP2 network, which has heterogeneous node strengths even with homogeneous link weights, confirms that asymmetry is driven by heterogeneity in node strengths rather than link weights.

Second, the heterogeneous circle network results demonstrate that, when there is heterogeneity in node strengths within the set of agents that comprise a maximal independent set, the existence of specialized equilibria now depends separately on the equilibrium conditions of each inactive agent. In particular, the pivotal agent is the one who is least structurally exposed to positive (effective) externalities. Agents with higher node strengths would be content to exit the contest at lower levels of α, but are forced to remain active by the unwillingness of this pivotal agent to drop out. Further, in a network with multiple maximal independent sets and node strengths such that the threshold value of α for the pivotal agent in each set differs, there will be a different range of α for which each specialized equilibrium exists.

7 Conclusion

In this paper, we introduce and analyze a model of contests with identity-dependent externalities that are governed by a network. Our theoretical results simultaneously broaden the scope of traditional contest theory and extend the network games literature to a setting in which players have non-linear best replies. The fully general model allows for heterogeneous externalities, both positive and negative—stemming from the allocation of the prize—that impact the payoffs of all players directly connected to the winner of the contest. We establish the existence of Nash equilibria and characterize sufficient conditions for uniqueness. Moreover, we illustrate the properties of equilibria and derive intuitive comparative statics for the tractable case in which links are homogenous. For two broad classes of networks (regular and core-to-periphery), we provide closed-form results and show that the comparative statics align with the intuition from our motivating examples. Introducing additional heterogeneity across links while maintaining homogenous node strengths illustrates that many of the characteristics of equilibria in similar network structures are similar to the homogenous case. Likewise, relatively more tractable examples for the fully heterogeneous case illustrate the breadth of the model.

Our framework can serve as a basis for studying a wide range of competitive situations, whether between firms or other organizations, individuals connected in a social network, or lobbyists with preferences over a multi-dimensional policy space. From a methodological perspective, we provide a novel approach—using the fact that the network contest game is a BR-potential game—to derive sufficient conditions for the uniqueness of Nash equilibria. This approach could potentially be applied to other network games in which best response functions are non-linear and alternative methods cannot directly be applied.

There are several ways in which our research may be extended. Our theoretical framework is very general, however, we do not allow for externalities to be asymmetric between pairs of agents. That is, in all of the networks, we assume that $g_{ij} = g_{ji}$ for every i, j. Nevertheless, relaxing this assumption may prove to be intractable unless accompanied by some additional structure. Another potential extension to the model might be to allow for payoff externalities to travel beyond the winner's immediate neighborhood, but with diluted impact proportional to the distance traveled. Finally, the intuitive comparative statics results we obtain for the more tractable cases examined in Section 4, where links are homogenous, can be examined empirically using a controlled laboratory experiment. In a separate paper (Boosey and Brown, 2022), we take up exactly this task, reporting the results of an experiment in which we systematically varied both the network and the size (and sign) of the externalities.

A Proofs

A.1 Proof of Theorem 1

We prove existence by applying Theorem 3.1 in Reny (1999). For completeness, we restate the theorem using our own notation.

Theorem (Theorem 3.1, Reny (1999))**.** *If $\Gamma = (X_i, \pi_i)_{i=1}^n$ is compact, quasiconcave, and better-reply secure, then it possesses a pure strategy Nash equilibrium.*

Let $\Gamma = (X_i, \pi_i)_{i=1}^n$ denote the normal-form of the network contest game. Note that while $X_i = \mathbb{R}_+$ for each $i \in N$, we can, without loss of generality, restrict the agents' strategies to compact subsets of $\mathbb{R}_+$. To see why, note that since $\alpha < 1$, $P_i \leqslant 1$, and $\sum_h g_{ih} \leqslant n - 1$, all strategies $x_i > 1 + (n - 1) = n$ are strictly dominated by $x_i = 0$. Thus, we can restrict the strategy sets to $\widehat{X}_i = [0, n]$, which is compact. Next, we note that each agent i's payoff function is concave, and thus also quasiconcave, in x_i. It remains to show that Γ is *better reply secure*. To do so, we first introduce some relevant definitions and another result by Bagh and Jofre (2006) that extends on Reny (1999).

Definition 4. In the game $\Gamma = (X_i, \pi_i)_{i=1}^n$, player i can *secure* a payoff of $\gamma \in \mathbb{R}$ at $x \in X$ if there exists $y_i \in X_i$ such that $\pi_i(y_i, \mathbf{x}'_{-i}) \geqslant \gamma$ for all $\mathbf{x}'_{-i}$ in some open neighborhood of $\mathbf{x}_{-i}$.

Definition 5. A game $\Gamma = (X_i, \pi_i)_{i=1}^n$ is *payoff secure* if for every $\mathbf{x} \in \mathbf{X}$ and every $\varepsilon > 0$, each player i can secure a payoff of $\pi_i(\mathbf{x}) - \varepsilon$ at $\mathbf{x}$.

Let $\Lambda = \{(\mathbf{x}, \pi) \in \mathbf{X} \times \mathbb{R}^n \,|\, \pi_i(\mathbf{x}) = \pi_i, \forall\, i\}$ denote the graph of the vector of payoff functions for the game and let $\overline{\Lambda}$ denote the closure of Λ in $\mathbf{X} \times \mathbb{R}^n$. Finally, define the frontier of Λ to be the set of points in $\overline{\Lambda}$ but not in Λ, denoted by $\mathrm{Fr}\Lambda = \overline{\Lambda} \backslash \Lambda$. The following definition is from Bagh and Jofre (2006).

Definition 6. A game $\Gamma = (X_i, \pi_i)_{i=1}^n$ is *weakly reciprocally upper semicontinuous* (wrusc) if, for any $(\mathbf{x}, \pi) \in \mathrm{Fr}\Lambda$, there is a player i and $\widehat{x}_i \in X_i$ such that $\pi_i(\widehat{x}_i, \mathbf{x}_{-i}) > \pi_i$.

Having defined payoff security and wrusc, we then appeal to the following result from Bagh and Jofre (2006).

Proposition 5 (Proposition 1, Bagh and Jofre (2006))**.** *If the game* $\Gamma = (X_i, \pi_i)_{i=1}^n$ *is payoff secure and wrusc, then it is better reply secure.*

To prove that Γ is payoff secure and wrusc, we follow a similar approach to Bagh and Jofre (2006) in their Example 3, which considers (a generalized form of) the standard contest game with Tullock (1980) contest success function.

(i) First, we show that the game is payoff secure. Note that payoffs are continuous except at $\mathbf{x} = \mathbf{0}$, where they are given by

$$\pi_i(\mathbf{0}) = \frac{1 + \alpha d_i}{n}$$

where $d_i = \sum_h g_{ih}$ is player i's weighted degree (or node strength) in the network. Then note that for $\tilde{x}_i > 0$, we have $\pi_i(\tilde{x}_i, \mathbf{0}) = 1 - \tilde{x}_i$, which is higher than $\pi_i(\mathbf{0})$ if $\tilde{x}_i < (n - 1 - \alpha d_i)/n$. Since $d_i \leqslant n - 1$ and $\alpha < 1$, the right hand side is strictly positive, so that such a $\tilde{x}_i > 0$ can be found. Then, since $\pi_i(\cdot)$ is continuous at $(\tilde{x}_i, \mathbf{0})$, there is a neighborhood V of $\mathbf{x}_{-i} = \mathbf{0}$ such that $\pi_i(\tilde{x}_i, \mathbf{x}'_{-i}) > \pi_i(0, \mathbf{0})$ for all $\mathbf{x}'_{-i} \in V$. Thus, the game is payoff secure at the point $\mathbf{x} = \mathbf{0}$. Payoff security at all other $\mathbf{x}$ is straightforward.

(ii) Second, we show that the game is wrusc. In this game (as in the standard contest game), the only points in $\mathrm{Fr}\Lambda$ must be points of the form $(\mathbf{0}, \pi)$ where

$\pi_i = \lim_{\mathbf{x}^k \to \mathbf{0}} \pi_i(\mathbf{x}^k)$ for all i. Note that

$$\begin{aligned}\sum_{i=1}^{n} \pi_i(\mathbf{x}^k) &= \sum_{i=1}^{n} P_i(\mathbf{x}^k) - \sum_{i=1}^{n} x_i + \alpha \sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij} P_j(\mathbf{x}^k) \\ &= 1 - \sum_{i=1}^{n} x_i + \alpha \sum_{i=1}^{n} d_i P_i(\mathbf{x}^k) \\ &\leqslant 1 - \sum_{i=1}^{n} x_i + \alpha(n-1)\end{aligned}$$

where the inequality follows from the fact that $d_i \leqslant n-1$ for all i and $\sum_{i=1}^{n} P_i(\mathbf{x}^k) = 1$. As such, $\lim_{\mathbf{x}^k \to \mathbf{0}} \sum_{i=1}^{n} \pi_i(\mathbf{x}^k) \leqslant 1 + \alpha(n-1)$ and thus, there exists some i for whom

$$\pi_i \leqslant \frac{1+\alpha(n-1)}{n}.$$

Notice that $\lim_{x_i \to 0} \pi_i(x_i, \mathbf{0}) = 1$. Thus, there exists $\widehat{x}_i > 0$ such that $\pi_i(\widehat{x}_i, \mathbf{0}) > \pi_i$, because $\alpha < 1$ ensures that $(1 + \alpha(n-1))/n < 1$. It follows that the game is wrusc.

Together, payoff security and wrusc imply better reply security, and applying Theorem 3.1 from Reny (1999), there exists a pure strategy Nash equilibrium. ☐

A.2 Proof of Proposition 1

Both parts of the proposition follow directly. From condition (i) in Lemma 1, it follows from the fact that $g_{ij} = 0$ for all $i, j \in A$ in a specialized equilibrium, that $x_i = \sum_{j \in A} x_j - (\sum_{j \in A} x_j)^2$ for all $i \in A$, which implies that all active players must be choosing the same investment $\bar{x}_A = \frac{n_A - 1}{n_A^2}$. Therefore, total investment is given by $X_A = \sum_{j \in A} x_j = (n_A - 1)/n_A$. Then, for the second condition in Proposition 1 to be satisfied, it must be the case that for all $i \in N - A$,

$$\frac{n_A - 1}{n_A^2}(n_A - \alpha d_A^i) \leqslant \frac{(n_A - 1)^2}{n_A^2}$$

$$\iff \qquad \alpha \begin{cases} \geqslant \frac{1}{d_A^i} & \text{if } d_A^i > 0 \\ \leqslant \frac{1}{d_A^i} & \text{if } d_A^i < 0 \end{cases}$$

Note that since $\alpha \geqslant 0$, the condition can never be satisfied if there exists some inactive agent, $k \in N - A$ for whom $d_A^k \leqslant 0$. Thus, a specialized equilibrium requires that $d_{N-A,A}$, defined to be the minimum of d_A^i over all $i \in N - A$, must be strictly positive, to ensure that the inequality is satisfied for all inactive players.

□

A.3 Proof of Proposition 3

Suppose that $A = N$ (that is, all agents are active). From Lemma 1, only condition (i) needs to be satisfied. Summing equation [5] for all n active players and rearranging gives

$$(n-1)\sum_{i=1}^{n} x_i - \alpha \sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}x_j \quad = \quad n\left(\sum_{i=1}^{n} x_i\right)^2$$

and positing $x_i = x$ for all i yields

$$\begin{aligned}(n-1)nx - \alpha nkx &= n(nx)^2 \\ (n-1) - \alpha k &= n^2 x\end{aligned}$$

from which x^* follows. □

A.4 Proof of Proposition 4

Suppose both types are active and consider condition (i) from Proposition 1. For each peripheral player, equation [5] reduces to

$$(n_c - \alpha)x_c + (n_c m - 1)x_p = (n_c x_c + n_c m x_p)^2$$

while for each core player, it simplifies to

$$(n_c - 1)(1-\alpha)x_c + (n_c m - \alpha m)x_p = (n_c x_c + n_c m x_p)^2.$$

From this, we obtain $x_c(1 + \alpha(n_c - 2)) = (1 - \alpha m)x_p$.

Substituting into the condition for the core players and solving yields the solution $x_c = (1 - \alpha m)\Delta$ and $x_p = (1 + \alpha(n_c - 2))\Delta$, where

$$\Delta \quad = \quad \frac{n_c\left[1 + m + \alpha m(n_c - 3)\right] - \left[1 + \alpha(n_c - 1 - \alpha m)\right]}{n_c^2\left[1 + m + \alpha m(n_c - 3)\right]^2} \geqslant 0.$$

For x_c to be strictly positive, we must have $\alpha < \frac{1}{m}$. Thus, a semi-symmetric equilibrium with full participation exists only when α is not too large. Once $\alpha \geqslant \frac{1}{m}$, there is a semi-symmetric equilibrium which is also a *specialized equilibrium* in which the core players are all inactive, while the peripheral players, who form a maximal independent set, invest the standard equilibrium investment for a contest between $n_c m$ individuals. $\square$

A.5 Proof of Lemma 2

We proceed by cases. Fix a player i.

Case 1. Suppose $\mathbf{x}_{-i}$ has at least two strictly positive components. Then, for any x_i, $A(x_i, \mathbf{x}_{-i}) \geqslant 2$. It follows from [8] that

$$\frac{\partial \mathbf{P}}{\partial x_i} = \sum_{h \neq i} (1 - \alpha g_{ih}) x_h - X_{tot}^2$$
$$\frac{\partial^2 \mathbf{P}}{\partial x_i^2} = -2X_{tot} < 0.$$

It follows that $x_i \in \arg\max \mathbf{P}(x_i, \mathbf{x}_{-i})$ if and only if

$$x_i \Big(\sum_{h \neq i} (1 - \alpha g_{ih}) x_h - X_{tot}^2 \Big) = 0$$

which implies

$$x_i = \max \left\{ 0, \sqrt{\sum_{h \neq i} (1 - \alpha g_{ih}) x_h} - \sum_{h \neq i} x_h \right\},$$

which is exactly the best response function $f_i(\mathbf{x}_{-i}, \alpha, \mathbf{G})$ derived in [4].

Case 2. Next, suppose $x_j > 0$ is the only positive component of $\mathbf{x}_{-i}$. From [8],

$$x_i > 0 \Rightarrow \mathbf{P}(x_i, \mathbf{x}_{-i}) = x_i x_j (1 - \alpha g_{ij}) - \frac{1}{3}(x_i + x_j)^3$$

whereas

$$x_i = 0 \Rightarrow \mathbf{P}(x_i, \mathbf{x}_{-i}) = -\frac{1}{3} x_j \big[\max_{h \neq j} (1 - \alpha g_{hj}) \big]^2.$$

Taking the limit as x_i approaches zero from above, we have $\lim_{x_i \to 0} \mathbf{P}(x_i, \mathbf{x}_{-i}) = -\frac{1}{3} x_j^3$, which is strictly greater than $\mathbf{P}(0, \mathbf{x}_{-i})$ if and only if

$$x_j < \max \big(1 - \alpha g_{ij}\big).$$

Multiplying through by x_j, player i's best response is interior at some $x_i > 0$ if and only if

$$x_j^2 < \max \big(1 - \alpha g_{ij}\big) x_j,$$

and is $x_i = 0$ otherwise, which again coincides with the best response function in [4].

Case 3. Finally, suppose $\mathbf{x}_{-i} = \mathbf{0}$. If $x_i > 0$, then

$$\mathbf{P}(x_i, \mathbf{0}) = -\frac{1}{3} x_i \big[\max_{h \neq i} (1 - \alpha g_{ih}) \big]^2,$$

which approaches zero (from below) as x_i approaches zero from above. In contrast, $x_i = 0$ implies $\mathbf{P}(\mathbf{0}) = -\frac{1}{3}\frac{(n-1)}{n} < 0$. As such, a maximizer does not exist for $\mathbf{P}$, just as the best response function for π_i is empty when $\mathbf{x}_{-i} = \mathbf{0}$.

By means of the three cases, we have verified that for an arbitrary player i, the set of maximizers for $\mathbf{P}$ given any $\mathbf{x}_{-i}$ coincide with the best responses according to the payoff functions π_i. Thus, $\mathbf{P}$ is a BR-potential for Γ. $\square$

A.6 Proof of Theorem 2

The network contest game is a best-response potential game (Voorneveld, 2000). Lemma 2 provides a BR-potential for the game, $\mathbf{P}$. Then, by Proposition 2.2 of Voorneveld (2000), the profile $\mathbf{x}$ is a Nash equilibrium of the network contest game if and only if it maximizes the BR-potential, $\mathbf{P}$. The remainder of the proof establishes conditions under which a unique maximizer exists for $\mathbf{P}$.

Recall that $\mathbf{P}$ is strictly concave if $\nabla^2 \mathbf{P}$ is negative definite. Before deriving the Hessian for $\mathbf{P}$, note that we can restrict the search for maxima to investment profiles $\mathbf{x}$ with $|A(\mathbf{x})| \geqslant 2$, since we have already established that there are no Nash equilibria in which fewer than 2 players are active. Thus, for any such $\mathbf{x}$, the diagonal elements of the Hessian $\nabla^2 \mathbf{P}$ are given by

$$\frac{\partial^2 \mathbf{P}}{\partial x_i^2} \quad = \quad -2 \sum_{h=1}^{n} x_h$$

while the cross-partial terms are symmetric and given by

$$\frac{\partial^2 \mathbf{P}}{\partial x_i \partial x_j} = \frac{\partial^2 \mathbf{P}}{\partial x_j \partial x_i} \quad = \quad (1 - \alpha g_{ij}) - 2 \sum_{h=1}^{n} x_h.$$

Rewriting in matrix form and using $X_{tot} = \sum_h x_h$ gives

$$\nabla^2 \mathbf{P} \quad = \quad \big(1 - 2X_{tot}\big)\mathbf{J} - [\mathbf{I} + \alpha \mathbf{G}],$$

where $\mathbf{J}$ denotes the $n \times n$ matrix of ones. Note that even if $\mathbf{I} + \alpha\mathbf{G}$ is positive definite, if $X_{tot} < 0.5$ and is small enough, the Hessian need not be negative definite. Our approach to getting around this problem is to partition the domain into two subsets, $\mathbf{X}^H$ and $\mathbf{X}^L$.

(i) If we restrict the domain of $\mathbf{P}$ to the set $\mathbf{X}^H$ of vectors $\mathbf{x}$ such that $X_{tot} \geqslant 0.5$, it is readily verified that $\mathbf{P}$ is strictly concave on the restricted domain if $\mathbf{I} + \alpha\mathbf{G}$ is positive definite, which is equivalent to the condition that $\alpha < 1/|\lambda_{min}(\mathbf{G})|$.

Before turning to Case (ii), suppose all of the links in the network are negative, as for part (a) of the Theorem. By Lemma 1, we must have for each active agent $i \in A$,

$$x_i + \sum_{j \in A} \alpha g_{ij} x_j = X_{tot}(1 - X_{tot}),$$

so that, using the fact that each $g_{ij} \leqslant 0$ and summing over all $i \in A$, we obtain

$$X_{tot} \geqslant n_A X_{tot}(1 - X_{tot}),$$

which simplifies to $X_{tot} \geqslant 1 - 1/n_A$. Then, because $n_A \geqslant 2$ in any equilibrium, we conclude that for the case where all links are negative, we must have $X_{tot} \geqslant 0.5$. This establishes part (i) of Theorem 2.

(ii) Next, consider the subdomain $\mathbf{X}^L$, which is composed of strategy profiles $\mathbf{x}$ with $X_{tot} < 0.5$. We proceed by direct argument. Suppose that $\mathbf{x}$ is a Nash equilibrium with $X_{tot} < 0.5$. First, by the argument above, this can not be the case if all links are negative—there must be at least one strictly positive link. We consider two cases:

(a) First, if there is any inactive player, k, we must have

$$X_{tot} - \alpha \sum_{h=1}^{n} g_{kh} x_h \quad \leqslant \quad \left(X_{tot}\right)^2.$$

When $\sum_{h=1}^{n} g_{ih} x_h > 0$, we can rearrange the inequality above to

$$\alpha \quad \geqslant \quad \frac{X_{tot}\left(1 - X_{tot}\right)}{\sum_{h=1}^{n} g_{kh} x_h},$$

and since $g_{kh} \leqslant 1$ for all k, h, it follows that $\alpha \geqslant 1 - X_{tot} > 0.5$. On the other hand, when $\sum_{h=1}^{n} g_{ih} x_h \leqslant 0$, we obtain

$$X_{tot} - \alpha \sum_{h=1}^{n} g_{kh} x_h \geqslant X_{tot} > (X_{tot})^2$$

which implies that k will not wish to remain inactive, contradicting that the profile is an equilibrium. Thus, if there is an equilibrium with an inactive player, such that $X_{tot} = \sum_h x_h < 0.5$, it must be the case that $\alpha > 0.5$.

(b) Second, suppose there is no inactive player for $\mathbf{x}$ with $X_{tot} < 0.5$. Then, for all n players, we must have

$$x_i + \alpha \sum_{h=1}^{n} g_{ih} x_h \quad = \quad X_{tot}(1 - X_{tot}).$$

Summing over all i, obtain

$$\begin{aligned}
& \alpha \sum_{i=1}^{n} \sum_{h=1}^{n} g_{ih} x_h = X_{tot}(n(1 - X_{tot}) - 1) \\
\Rightarrow \quad & \alpha \sum_{h=1}^{n} x_h \sum_{i=1}^{n} g_{ih} > X_{tot} \left(\frac{n-2}{2} \right) \\
\Rightarrow \quad & \alpha \sum_{h=1}^{n} d_h x_h > X_{tot} \left(\frac{n-2}{2} \right) \\
\Rightarrow \quad & \alpha \Delta(\mathbf{G}) X_{tot} > X_{tot} \left(\frac{n-2}{2} \right) \\
\Rightarrow \quad & \alpha > \frac{n-2}{2\Delta(\mathbf{G})},
\end{aligned}$$

where the second line follows from $1 - X_{tot} > 0.5$, and the fourth line from the fact that $\Delta(\mathbf{G})$ is the maximum degree of $\mathbf{G}$.

It follows that if $\alpha \leqslant 0.5$ and $\alpha \leqslant 0.5(n-2)/\Delta(\mathbf{G})$, there cannot be a Nash equilibrium in $\mathbf{X}^L$. By the existence of an equilibrium, there must exist at least one equilibrium in $\mathbf{X}^H$. If we also have that $\alpha < 1/|\lambda_{min}(\mathbf{G})|$, then $[\mathbf{I} + \alpha \mathbf{G}]$ is positive definite, $\mathbf{P}$ is strictly concave on $\mathbf{X}^H$, and there exists a unique Nash equilibrium, $\mathbf{x} \in \mathbf{X}^H$, such that $X_{tot} \geqslant 0.5$. This establishes part (ii) of Theorem 2. ☐

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