Notation and definitions

In this section, I discuss the definitions of a (non-)collegial voting rule, pivotal coalitions, pivotal gradient restrictions, and the strongly stable core so that I can state the results in a rigorous manner. All the definitions that I use, except that of a strongly stable core, are Schofield's original definitions. I begin with the formal notation.

$N = \{1,2,\ldots n\}$, the (finite) set of all voters,
$W \subseteq R^w$, the set of alternatives,

$v = (v_1,v_2,\ldots v_n.)'$ where v_i denotes the number of votes voter *i* has, $v_i \geq 0 \; \forall \, i$, and
$v_i \geq v_{i+1} \; \forall \, i \in N$,

$V = \sum_{i=1}^{n} v_i$, the total number of votes.

The set of alternatives, W, is a subset of Euclidean space of dimension w. I assume that the set is convex and compact, and the interior of W in R^w is nonempty. I also assume that individual voters have type I Euclidean preferences unless I state otherwise. The definition of v above clearly shows that the voting game is weighted, that is, voters can have different numbers of votes, with voter I being the strongest voter (i.e., the one with the most votes), voter 2 being the second strongest voter, and so on. Now I state the necessary definitions.

Definition 1. Let *D* be the set of decisive coalitions and σ be a voting rule. "[D]efine $\kappa(D_\sigma)$, *the collegium of* σ, to be that subset of N that belongs to every decisive coalition. If $\kappa(D_\sigma)$ is non-empty, then call σ *collegial* ... If $\kappa(D_\sigma)$ is empty, then call σ *non-collegial"* (Schofield, 1986:270).

That is, when the voting rule is collegial, there is one voter or a set of voters that belongs to every winning coalition. Since any winning coalition requires this voter (or this set of voters), he (or they) has the power to veto any alternative even if everybody else wants it. When the voting rule is non-collegial, there is no such voter or such set of voters present. Obviously the voting game is more democratic when it is non-collegial.

Definition 2. "Given a family, *D*, of subsets of N, and any set $L \subseteq N$, define the set of *pivotal coalitions for D* in L, written $E_L(D)$, as the set of coalitions $M \subseteq L$, such that for every binary partition, {R,S} of L-M either $M \cup R$ or $M \cup S$ belongs to *D*" (Schofield, 1986:276). That is, a coalition, M, is defined "to be *pivotal* in a subset L of the voters, if it is the case that whenever we partition L-M into two subsets, at least one of these subsets, together with the members of M, constitutes a decisive coalition" (McKelvey and Schofield, 1987:923).

Definition 3. "For any $x \in W$, and $i \in N$, let $p_i(x) = \nabla u_i(x) \in R^w$ represent voter *i*'s utility gradient at the point x" (McKelvey and Schofield, 1987:925). Let $p = (p_1,\ldots,p_n) \in (R^w)^n$

be a profile of utility gradients. "Say that p satisfies the *pivotal gradient restrictions* (PGR) with respect to *D* and L if and only if for every $M \in E_L(D)$, $0 \in p_{M^*}$. (the convex hull of $\{p_i: i \in M^*\}$) such that $M^* = \{i \in L: p_i \in sp_M\}$, and sp_M is the set of vectors in R^W spanned by $\{p_i: i \in M\}$. Say that p satisfies PGR with respect to *D* if and only if p satisfies PGR with respect to *D* and L, for every $L \subseteq N$" (Schofield, 1986:276). That is, "[fjor every pivotal coalition M in L, the set of utility gradients which lie in the subspace spanned by those in M, must positively span 0 (the zero vector) ... Say that the pivotal coalition, $M \in E_L$ is 'blocked' if $0 \in p_{M^*}.(x)$. If M is blocked, then there are some members of L, whose gradients lie in the same subspace as those of M, but *not* in the same half space ... Thus, the members of M* cannot agree on any common direction to move. The PGR condition, then, simply specifies that every pivotal coalition, in every subset L of N, must be blocked in the above sense" (McKelvey and Schofield, 1987:923-926).

McKelvey and Schofield showed that, with semi-convex preferences, PGR is a necessary and sufficient condition for an interior point of W to belong to the core (McKelvey and Schofield, 1987:928).

Now I define the strongly stable core.

Definition 4. An alternative, x, belongs to the *Strongly Stable Core* if and only if (i) the core is nonempty at *u* and x belongs to the core where $u = (u_1 \ldots u_n)$ is a profile of smooth utility functions for each member of the society; and (ii) the core at û is still nonempty and x still belongs to core for any sufficiently small perturbation, û of u, where, in case there exists *j* with $p_j(x) = 0$, I further assume that û is of the form, $\hat{u} = (\hat{u}_1,\ldots,\hat{u}_{j-1},u_j,\hat{u}_{j+1},\ldots,\hat{u}_n)$. Here I measure the closeness of û and u using the uniform topology, that is, û is close to *u* if and only if $\hat{u}_i$ is close to u_i for all $i \in N$.

The following descriptive definition of a Strongly Stable Core is equivalent to the one given above. An alternative belongs to the Strongly Stable Core when (i) no winning coalition can agree on changing to some alternative outcome; and (ii) under any sufficiently small change in the preferences (with an exception noted above), (i) still holds. (i) makes this outcome a core and (ii) makes it strongly stable.

The formal definition above clearly reflects the differences between the strongly stable core (called SSC hereafter) and the structurally stable core (SC). Mainly, for an alternative to belong to SSC, it has to belong to the core both before and after perturbing voter preferences, and if some voter's ideal point belongs to the core, then that voter's utility function is not perturbed to get the strongly stable core.