Choosing Your Own Luck: Strategic Risk Taking and Effort in Contests*

Kyungmin Kim R. Vijay Krishna Dmitry Ryvkin

(Preliminary Draft)

September 25, 2023

Abstract

We consider the problem of optimal contest design in an environment where contestants choose not only their effort, but also the distribution of shocks affecting their output. We show that the presence of such strategic risk taking has a stark effect on contest design: The winner-take-all contest, whereby the entire prize budget is allocated to the top performer, maximizes the expected effort (or output) of the agents *regardless of the shape of their cost of effort*.

JEL Classification Codes: C72, C73

Keywords: contest; strategic risk taking; effort; prize allocation

*Kim: Department of Economics, Emory University (email: kyungmin.kim@emory.edu); Krishna: Department of Economics, Florida State University (email: rvk3570@gmail.com); Ryvkin: School of Economics, Finance and Marketing, RMIT University; and Department of Economics, Florida State University (email: dmitry.ryvkin@rmit.edu.au, dryvkin@fsu.edu);

1. Introduction

Contests—allocation mechanisms based on ordinal performance comparisons—are used extensively to motivate agents in organizations and other settings.[1] A central question in the theoretical literature on contests has been that of optimal contest design: How should a fixed budget be distributed across performance ranks? In particular, how does increasing prize inequality affect agents' effort? In this paper, we address this question in a novel environment where each player not only chooses costly effort, but also engages in strategic risk taking.

A recent contribution by Fang, Noe, and Strack (2020) (henceforth, FNS) offers a point of departure for this paper. Their results relevant to our analysis can be summarized as follows.[2]

Theorem 0 (FNS)**.** *In the all-pay contest without risk taking, as prizes become more unequal (in the majorisation order, defined below), the (symmetric) equilibrium expected effort rises if effort costs are concave and falls if effort costs are convex.*

In other words, FNS show that prize inequality encourages agents' efforts if their costs are concave but does precisely the opposite if costs are convex; for completeness, we reproduce their key argument in Section 3. Among other things, this implies that the equilibrium expected effort is maximised by the winner-take-all (WTA) contest if effort costs are concave, and by the punish-the-bottom (PTB) contest—rewarding equally all players except the worst performer—if effort costs are convex.[3]

We revisit the contest design problem in a novel environment where each player not only exerts effort but also "chooses their own luck." Specifically, as in the standard all-pay contest (eg, Hillman and Riley, 1989; Siegel, 2009; Fang, Noe, and Strack, 2020), each player selects his effort x_i at cost $c(x_i)$. Concurrently, he also chooses an *arbitrary* unbiased random noise ε_i, so that the final (realised) output is a nonnegative random variable $Y_i = x_i + \varepsilon_i \geqslant 0$. This latter element of the model is our main (sole) innovation relative to the standard all-pay contest, but it is not entirely new to the literature. In fact, that part is just as in the literature on *pure* risk-taking contests where each player chooses a distribution with a fixed mean (eg, Myerson, 1993; Ray and Robson, 2012;

[1] Examples include promotions and bonuses (Bognanno, 2001; Baker, Jensen, and Murphy, 1988), sales contests (Lim, Ahearne, and Ham, 2009), forced ranking systems (Bretz Jr., Milkovich, and Read, 1992), and R&D competition (Terwiesch and Ulrich, 2009).

[2] To our knowledge, FNS provide the most comprehensive results for complete-information contests without noise.

[3] In their paper, FNS focus on the convex cost case and, therefore, argue that "turning up the heat" (ie, increasing prize inequality) is detrimental to effort. It is also clear from their analysis that when costs are neither (globally) concave nor convex, the result would be more nuanced and sensitive to the shape of the cost function. Indeed, it is possible that the optimal allocation is neither WTA nor PTB.

Ke et al., 2021; Fang and Noe, 2022). Our contribution in modelling is to bring together two distinct contest models and study how strategic risk taking influences agents' effort choices.[4]

Our model of strategic risk taking can be interpreted in several ways. First, it arises when each player can flexibly engage in fair gambling. Importantly, this interpretation applies not only when both x_i and Y_i correspond to monetary values, but also when they represent physical values of a good: x_i captures the vertical/intrinsic quality of a good, while ε_i represents the horizontal/design aspect of a good. Second, our model captures a situation where an agent's output depends not just on his effort, but also on potentially many creative activities or decisions with uncertain outcomes. For example, in an innovation contest researchers choose which directions to pursue, which methods to use to pursue them, or when to stop experimenting. In an architectural contest, there is a choice among a large number of possible styles and designs with many minute details pertaining to each. In these settings, agents' (rich) choices cannot be summarized by a scalar value (*effort*). Our model captures such a situation in a simple manner. Yet another interpretation of endogenous risk taking is signal jamming. Indeed, as in a standard moral hazard environment, agents may be interested in obfuscating their true effort with nonproductive activities such as self-promotion, engaging in a form of (reverse) Bayesian persuasion. The assumption that noise is mean-preserving then serves as a disciplining constraint similar to the one used in the information design literature.

Our main result is as follows.

Main Theorem. *In our model of strategic risk taking, the equilibrium expected effort (and output) is maximised when the prize schedule is winner-take-all, regardless of the shape of the cost function c (satisfying mild technical assumptions).*

The Main Theorem is a comparative static result regarding the *equilibrium* expected effort. As illustrated shortly, a few key equilibrium properties hold regardless of the structure of the cost function, but some properties do depend on it. This makes it unavoidable to consider several different cases and establish the results for each case. Specifically, in the main text of the paper we consider in detail and prove the Main Theorem for four representative cases, with c (i) globally concave, (ii) globally convex, (iii) initially concave and then convex, and (iv) initially convex and then concave. The existing literature has restricted attention to (i) and (ii), not only because of their tractability but also because (i) and (ii) are enough for a nuanced result—namely, that the optimal contest depends on the structure of c. Yet, cost structures (iii) and (iv) are relevant for many applications, and the approach we develop in this paper allows us

[4]Hvide (2002) and Gilpatric (2009) also consider a model in which each player chooses both effort and risk. However, unlike us, they make use of structural assumptions on risk taking, restricting the set of possible distributions of Y_i (ie, the set of possible *joint* distributions of (x_i, ε_i)).

to consider them in a unified manner. Moreover, these techniques can be extended to more general cost functions with multiple inflexion points, whose analysis is relegated to Appendix D.

The key observation for our equilibrium analysis, and in establishing the Main Theorem, is that strategic risk taking reduces players' effort costs to produce a stochastic output Y_i in a way that their *virtual* (effective) cost function of output, ξ^*, is concave, regardless of the shape of the underlying effort cost function c. To see this, suppose a player wishes to produce output y_1 or y_2 with equal probability. If c is concave then randomising over efforts—exerting effort $x_i = y_1$ and $x_i = y_2$ each with probability $1/2$—is more economical than deterministically choosing effort $x_i = (y_1 + y_2)/2$ and then randomising over outputs by choosing $\varepsilon_i = \pm|y_2 - y_1|/2$ with equal probability. In this case, strategic risk taking is irrelevant, and $\xi^* = c$. Conversely, if c is convex then randomising over outputs is more economical than randomising over effort, so the player chooses a deterministic effort x_i and then randomizes over outputs. In this case, the resulting virtual cost ξ^* is linear because at the risk-taking stage, the player faces the mean constraint that $\mathbf{E}[Y_i] = x_i$ and ξ^* reflects the corresponding shadow cost. This basic idea applies to the entire relevant region of c and also irrespective of the structure of c. Therefore, the virtual cost function ξ^* is always concave.[5]

The above observation provides a bridge between the existing result for the case of concave costs—whereby an increase in prize inequality raises expected effort—and our Main Theorem: The WTA contest maximises the expected effort regardless of the shape of c because the virtual cost function ξ^* is always concave. If ξ^* were independent of the prize allocation then the Main Theorem would follow immediately from the concavity of ξ^* and Theorem 0. However, ξ^* is itself an equilibrium object that depends on prize allocation; and how it does so is crucial for the comparative statics underlying the Main Theorem. Our main technical challenge is, therefore, to characterize how ξ^* depends on prize inequality and, in turn, how ξ^* affects the equilibrium effort.

Finding the optimal (effort-maximising) prize allocation is a classical problem in the literature on contests. The latest and most comprehensive treatments (using three different contest models) are by Moldovanu and Sela (2001) for incomplete-information contests with private types; by Drugov and Ryvkin (2020) for complete-information contests with exogenous noise à la Lazear and Rosen (1981); and by FNS for complete-information contests without noise. In all three environments, the results are nuanced; the WTA contest can be optimal, but more equal prizes can also be optimal, including

[5]This result resembles a canonical result in the literature on risk-taking contests, namely, that in equilibrium each player should face a weakly concave value function; otherwise, those players facing convex value functions would take extreme risks, unraveling the equilibrium. The result still appears in our model (see Section 3), but it is distinct from the reason why ξ^* is concave. In our model, concavity of ξ^* follows from each player's (individual) cost minimization, not because it is necessary to provide proper risk-taking incentives for players.

extreme prize sharing in the form of PTB. Importantly, in all these models the results are sensitive to the shape of effort costs, and cost functions beyond globally concave or convex have not been considered.

The two most closely related papers to ours are those on the effects of prize allocation on flexible risk taking in the absence of effort (ie, with an exogenous mean output). Fang and Noe (2022) consider a principal facing a selection problem: heterogeneous contestants compete for promotion by flexibly selecting stochastic output as a mean-preserving spread of their ability. The authors show that less competitive promotion policies—effectively, more equitable prize schedules—reduce risk taking and lead to improved selection in equilibrium. These results are echoed by Ke et al. (2021) who show, both theoretically and experimentally, that increasing prize inequality leads to more dispersion in output.

Methodologically, we leverage the technical results by Dworczak and Martini (2019). As illustrated in Section 4, the problem of finding the cost-minimising effort distribution for a given distribution of output is mathematically identical to the Bayesian persuasion problem with a continuous state space studied by Dworczak and Martini (2019). We use their results to determine the structure of the virtual cost function ξ^*.

The rest of the paper is structured as follows. The model is formally set up in Section 2. In Section 3, we reproduce, for completeness, the arguments from FNS underlying Theorem 0 and heuristically establish the Main Theorem for two tractable cases (either globally concave or globally convex). In Section 4, we provide the key reformulation of the model as a virtual contest. In Section 5, we return to the special case from Section 3 and show how the results for convex costs are obtained as a straightforward corollary. In Sections 6 and 7, we discuss more complex representative cases—concave-convex and convex-concave costs, relegating the analysis of more general cost functions to Appendix D. Section 8 concludes.

2. The Model

We build upon the standard all-pay contest. There are n $(\geqslant 2)$ players, each choosing *effort* $x_i \in \mathbb{R}_+$ according to the common cost function $c \in \mathbb{R}_+^{\mathbb{R}+}$. We assume that c is strictly increasing and twice continuously differentiable, satisfies $c(0) = 0$, and reaches 1 at a finite value of effort. Reflecting the possibility of mixing, we represent each player i's choice of effort as a non-negative random variable X_i. The associated expected cost of effort is given by $\mathbf{E}[c(X_i)]$.

Strategic risk taking is modelled as follows: Concurrently with effort X_i, each player i chooses a random variable ε_i leading to *output* $Y_i = X_i + \varepsilon_i$, subject to two constraints: (i) $\mathbf{E}[\varepsilon_i|X_i] = 0$, and (ii) $Y_i \geqslant 0$ almost surely. In other words, each player can add any unbiased noise to X_i, as long as the resulting output Y_i is non-negative. By definition, Y_i is feasible from X_i if, and only if, Y_i is a non-negative mean-preserving spread of X_i. Such a pair (X_i, Y_i) is said to *admissible*. As usual, we use X_{-i} and Y_{-i} to denote strategy profiles excluding player i's.

A *contest* is defined by a vector $\mathbf{v} = (v_1, ..., v_n) \in \mathbb{R}^n_+$, where v_k represents the prize to the player who produces the k-th highest output. We assume that prizes are monotonically decreasing in rank, and the total prize budget is normalised to one. In addition, because setting $v_n > 0$ (ie, giving surplus to the worst performer) is always detrimental to players' incentives, we restrict attention to the prize vectors such that $v_n = 0$. Let $\mathscr{V} := \{\mathbf{v} \in \mathbb{R}^n_+ : v_1 \geqslant \ldots \geqslant v_n = 0, \sum_{k=1}^n v_k = 1\}$ denote the set of all prize vectors (contests) that satisfy these restrictions. The usual winner-take-all (WTA) contest corresponds to $\mathbf{v}^{\text{WTA}} = (1, 0, ..., 0)$, while the "punish-the-bottom" (PTB) contest has $\mathbf{v}^{\text{PTB}} = (\frac{1}{n-1}, \ldots, \frac{1}{n-1}, 0)$.

Given $\mathbf{v} \in \mathscr{V}$, a player's payoff is given by

$$u_i(X_i, Y_i, X_{-i}, Y_{-i}) = \sum_{k=1}^{n} v_k \cdot \mathbf{P}[Y_i = Y_{(k:n)}] - \mathbf{E}[c(X_i)],$$

where $Y_{(1:n)} \geqslant \ldots \geqslant Y_{(n:n)}$ represent the order statistics of $(Y_1, ..., Y_n)$. For notational simplicity, we ignore ties, which will arise with probability 0 in equilibrium.

As usual, a Nash equilibrium is a profile of admissible effort-output combinations, $(X_i^*, Y_i^*)_{i=1}^n$, such that $u_i(X_i^*, Y_i^*, X_{-i}^*, Y_{-i}^*) \geqslant u_i(X_i, Y_i, X_{-i}^*, Y_{-i}^*)$ for all admissible (X_i, Y_i) and $i = 1, \ldots, n$. Following the literature, we focus on symmetric equilibria and use (X^*, Y^*) to denote a symmetric equilibrium strategy.

Inequality order. As in a few recent studies (Vojnović, 2015; Fang, Noe, and Strack, 2020; Drugov and Ryvkin, 2020), we adopt the *majorisation order* over $\mathscr{V}$ to compare prize schedules in terms of the level of inequality.[6] For $\mathbf{v}, \mathbf{w} \in \mathscr{V}$, $\mathbf{w}$ *majorises* $\mathbf{v}$—or, $\mathbf{w}$ is *more unequal* than $\mathbf{v}$—if $\sum_{i=1}^k (w_i - v_i) \geqslant 0$ for all $k = 1, \ldots, n$. Clearly, $\mathbf{v}^{\text{WTA}}$ majorises any $\mathbf{v} \in \mathscr{V}$, while $\mathbf{v}^{\text{PTB}}$ is majorised by any $\mathbf{v} \in \mathscr{V}$. Therefore, $\mathbf{v}^{\text{WTA}}$ is the most unequal contest, while $\mathbf{v}^{\text{PTB}}$ is the most equal contest, in $\mathscr{V}$. An elementary reduction in inequality is known as a *Pigou-Dalton (PD) transfer*, whereby given a $\mathbf{w} \in \mathscr{V}$, another prize schedule $\mathbf{v} \in \mathscr{V}$ is formed such that $v_i = w_i - \delta$ and $v_j = w_j + \delta$ for some $i < j$ and $\delta > 0$, and $v_k = w_k$ for all $k \neq i, j$. That is, a PD transfer reduces the prize to the i-th place and raises the prize to the j-th place by δ, with $i < j$. We will write this as $\mathbf{w} \succ_{ij}^{\delta} \mathbf{v}$ (where $i < j$ is implied). Importantly, if $\mathbf{w}$ majorises $\mathbf{v}$, then $\mathbf{v}$ can be obtained from $\mathbf{w}$ via a finite sequence of such PD transfers. Therefore, in many instances, in order to prove a comparative static result for the majorisation order it is sufficient to

[6] For a comprehensive discussion of the majorisation order and its applications, see, eg, Marshall, Olkin, and Arnold (2011).

prove it only for an arbitrary PD transfer.

3. First Pass: Prior Results and Elementary Analyses

In this section, we reproduce the main arguments of FNS pertaining to Theorem 0 and prove the Main Theorem for two simplest cases via elementary analyses.

3.1. Contests Without Risk Taking

Consider the standard all-pay contest in which each player's output is completely determined by his effort, ie, $Y_i = X_i$. As is well known, this contest has a unique symmetric (mixed strategy) equilibrium in which the distribution of effort F is continuous and supported on $[0, c^{-1}(v_1)]$, and all players earn zero rents (cf. Barut and Kovenock, 1998). To characterize the equilibrium, suppose a player exerts effort x, while all other players randomize according to F. The indicative player's payoff is given by $\Phi(F(x); \mathbf{v}) - c(x)$, where $\Phi(F(x); \mathbf{v})$ represents the player's expected winnings from the contest. The function $\Phi(\cdot; \mathbf{v}) : [0,1] \to \mathbb{R}_+$ can be written explicitly as

$$\Phi(q; \mathbf{v}) = \sum_{i=1}^{n} \binom{n-1}{i-1} q^{n-i}(1-q)^{i-1} v_i. \tag{3.1}$$

To understand the structure of $\Phi(q; \mathbf{v})$, suppose a player outperforms every other contestant with probability q (which corresponds to $F(x)$). In order to be ranked i-th and receive prize v_i, the player must be above $n-i$ players while also being below $i-1$ players. For a given set of other players' identities, the probability of this event is $q^{n-i}(1-q)^{i-1}$, and the binomial coefficient $\binom{n-1}{i-1}$ in [3.1] counts the number of ways the other players' identities can be selected. For any $\mathbf{v} \in \mathscr{V}$, $q \mapsto \Phi(q; \mathbf{v}) \in [0, v_1]$ is a continuous and strictly increasing function; thus, $\Phi^{-1}(t; \mathbf{v})$ is also a continuous and strictly increasing function of t over $[0, v_1]$.

That players earn zero rent in equilibrium implies that the symmetric equilibrium distribution of effort is given by

$$F(x) = \Phi^{-1}(\min\{c(x), v_1\}; \mathbf{v}), \qquad x \in \mathbb{R}_+. \tag{3.2}$$

In equilibrium, each player's expected winnings is $1/n$. The zero-rent condition now implies that each individual contestant's expected cost is equal to $1/n$, that is,

$$\frac{1}{n} = \int \Phi(F(x); \mathbf{v}) \, \mathrm{d}F(x) = \int c(x) \, \mathrm{d}F(x) \qquad \text{for all } \mathbf{v} \in \mathscr{V}. \tag{3.3}$$

It is clear from [3.2] that the equilibrium F depends on the prize distribution $\mathbf{v}$. In what follows, we will suppress this dependence unless different prize distributions are explicitly compared.

Consider two prize schedules $\mathbf{v}^1, \mathbf{v}^2 \in \mathscr{V}$ such that $\mathbf{v}^2 >_{ij}^{\delta} \mathbf{v}^1$. For each $i = 1, 2$, let $F(\cdot; \mathbf{v}^i)$ denote the equilibrium distribution under $\mathbf{v}^i$ and X_i denote the corresponding random variable. It can be shown that $\Phi(q; \mathbf{v}^2)$ crosses $\Phi(q; \mathbf{v}^1)$ once from below:[7] Intuitively, more prize inequality lowers (respectively, raises) the expected earnings from low (respectively, high) efforts. Together with [3.2], this implies that $F(x; \mathbf{v}^2)$ crosses $F(x; \mathbf{v}^1)$ once from above. If $\mathbf{E}[X^1] \geqslant \mathbf{E}[X^2]$ then X^1 dominates X^2 in the *increasing concave* order (see, eg, Theorem 4.A.22 of Shaked and Shanthikumar, 2007). Therefore, if c is strictly concave then $\mathbf{E}[c(X^1)] = 1/n > \mathbf{E}[c(X^2)]$, which contradicts [3.3]; this means that it must be that $\mathbf{E}[X^1] < \mathbf{E}[X^2]$. A similar argument shows that under convex costs, we must have $\mathbf{E}[X^1] > \mathbf{E}[X^2]$, thus implying Theorem 0.

Intuitively, if prizes become more unequal then players have an incentive to "swing for the fences," ie, take more risk, because higher values of effort—those more likely earning them top prizes—become relatively more profitable. When c is convex, this additional risk makes effort more costly. However, the expected cost of effort has to stay fixed in equilibrium (cf. the zero-rent condition [3.3]); hence, the expected effort goes down. By contrast, if c is concave then additional risk lowers overall effort costs, in which case the expected effort increases.

3.2. Strategic Risk Taking for Concave or Convex Costs

Next, consider our model in which each player chooses effort X_i and noise ε_i, so her output is given by $Y_i = X_i + \varepsilon_i$ with $\mathbf{E}[\varepsilon_i | X_i] = 0$ a.s.

Concave costs. Suppose c is strictly concave. Then, by Jensen's inequality, $\mathbf{E}[c(X_i)] \geqslant \mathbf{E}[c(Y_i)]$ whenever Y_i is a mean-preserving spread of X_i. This implies that taking strategic risk (ie, setting $X_i \neq Y_i$) only raises effort costs, so every player will choose $X_i = Y_i$. Since strategic risk taking is irrelevant, the problem reduces to the standard case summarized above, in which, by Theorem 0, the WTA contest delivers the greatest expected effort.[8]

[7] See Lemma A2 in FNS for a formal proof. For illustration, suppose $n = 3$, $\mathbf{v}^1 = (v_1, 1 - v_1, 0)$, and $\mathbf{v}^2 = (v_1 + \delta, 1 - v_1 - \delta, 0)$. Equation [3.1] then gives $\Phi(q; \mathbf{v}^2) - \Phi(q; \mathbf{v}^1) = [q^2 - 2q(1-q)]\delta = q(3q-2)\delta$, whose sign changes from negative to positive at $q = 2/3$.

[8] If costs are only *weakly* convex, then there may exist equilibria with risk taking. However, all those are outcome(output)-equivalent to those *without* risk taking. For example, if c is linear then there exist both an equilibrium with no risk taking and an equilibrium with no effort randomization; in fact, all mixtures between them, yielding the same output distribution, are also equilibria.

Convex costs. A similar argument shows that if c is strictly convex, an equilibrium necessarily entails a deterministic effort; that is, in equilibrium X_i is degenerate at some x_i. Let x_{d} denote the symmetric deterministic equilibrium effort. Given x_{d}, the game reduces to the standard risk-taking contest where each player chooses Y_i subject to $\mathbf{E}[Y_i] = x_{\mathrm{d}}$. Myerson (1993) provides a general characterization for the game and, in particular, shows that the unique symmetric equilibrium distribution G (of Y) satisfies

$$[3.4] \qquad \Phi(G(y); \mathbf{v}) = \min\left\{\frac{y}{nx_{\mathrm{d}}}, v_1\right\} \Rightarrow G(y) = \Phi^{-1}(\min\{y/(nx_{\mathrm{d}}), v_1\}; \mathbf{v}).$$

As with F, we explicitly indicate the dependence of the equilibrium G on $\mathbf{v}$ only as needed.

To identify the equilibrium effort x_{d}, suppose all other players select x_{d} and the distribution G in [3.4]. Then, an individual player's problem is

$$[3.5] \qquad \max_{x_i, G_i} \int \Phi(G(y); \mathbf{v})\, \mathrm{d}G_i(y) - c(x_i) \qquad \text{s.t. } x_i = \int y\, \mathrm{d}G_i(y).$$

As shown in [3.4], $\Phi(G(y); \mathbf{v})$ is concave in y, and so given x_i, player i has no incentive to take strategic risk. Thus, one (but not the only) solution to his problem in [3.5] is to take G_i to be degenerate at x_i. This reduces the problem to

$$\max_{x_i} \left[\min\left\{\frac{x_i}{nx_{\mathrm{d}}}, v_1\right\} - c(x_i)\right].$$

From here, it is straightforward that the symmetric equilibrium effort x_{d} is such that $x_{\mathrm{d}} c'(x_{\mathrm{d}}) = 1/n$.[9] It is then clear that (x_{d}, G), with G given by [3.4], is the unique symmetric equilibrium of the given contest with strategic risk taking.

This equilibrium has two important properties. First, the equilibrium effort x_{d} is independent of $\mathbf{v}$. Thus, contest design does not affect players' effort choices, implying that the WTA contest performs as well as any other contest in $\mathscr{V}$. This neutrality result is in stark contrast with the corresponding finding by FNS, namely, that if c is convex then more prize inequality disincentivises players and so the WTA contest minimises the expected effort. Second, players earn positive rents. Indeed, using symmetry we can write each player's equilibrium payoff as

$$\pi = \frac{1}{n} - c(x_{\mathrm{d}}) > \frac{1}{n} - x_{\mathrm{d}} c'(x_{\mathrm{d}}) = 0,$$

where the inequality follows from the strict convexity of c.

[9] There exists a unique value that satisfies $xc'(x) = 1/n$—so a unique symmetric equilibrium—because (i) $xc'(x)$ is strictly increasing and (ii) $xc'(x) \geqslant c(x) > 1/n$ at a finite x.

4. General Characterisation

The two cases considered in the previous section look far apart—in fact, opposite—from each other. However, they do share some important similarities. To see this, notice that in the convex case, given x_{d}, $\tilde{\xi}(y) := y/(nx_{\mathrm{d}})$ represents the shadow cost of producing y.[10] In the concave case, risk taking is irrelevant, so the shadow cost of producing y coincides with the corresponding effort cost (ie, $\tilde{\xi}(y) = c(y)$). In both cases, $\tilde{\xi}(y)$ is concave in y, and G is a symmetric equilibrium in a contest with cost $\tilde{\xi}(y)$ (see [3.2] and [3.4]). From this perspective, it is not surprising that the WTA contest maximises the expected effort whether the underlying cost function c is concave or convex. In this section, we show that a similar *concave* shadow cost $\tilde{\xi}$ arises for *any* cost function c.

4.1. Virtual Cost

Let F_i denote the distribution of effort X_i and G_i of output Y_i. In addition, we denote by MPS(F_i) the set of all non-negative mean-preserving *spreads* of F_i and by MPC(G_i) the set of all mean-preserving *contractions* of G_i. In what follows, we shall denote strategies by the respective distributions over effort and output, and a pair (F_i, G_i) is admissible if $F_i \in$ MPC(G_i), or equivalently, $G_i \in$ MPS(F_i).

By definition, (F^*, G^*) constitutes a symmetric equilibrium for the contest $\mathbf{v} \in \mathscr{V}$ if it solves

$$\text{[4.1]} \qquad \max_{F,G\in\Delta(\mathbb{R}_+)} \int \Phi(G^*(y); \mathbf{v})\,\mathrm{d}G(y) - \int c(x)\,\mathrm{d}F(x) \quad \text{s.t.} \quad F \in \mathrm{MPC}(G).$$

Notice that F affects only the second term in the objective function, so given $G = G^*$, [4.1] reduces to

$$\text{[4.2]} \qquad \min_{F\in\Delta(\mathbb{R}_+)} \int c(x)\,\mathrm{d}F(x) \quad \text{s.t.} \quad F \in \mathrm{MPC}(G^*).$$

This means that F^* must be a *cost-minimising* mean-preserving contraction of G^*. A crucial observation is that this cost-minimisation problem is isomorphic to the Bayesian persuasion problem studied by Dworczak and Martini (2019) (DM, hereafter).[11] Their results relevant to our analysis are translated and summarized in the following result.

Proposition 4.1. Let G^* be a cdf with a compact and convex support in $\mathbb{R}_+$. If F^* solves [4.2], then there exists a concave function $\xi^* \in \mathbb{R}^{\mathbb{R}_+}$ such that (i) $\xi^* \leqslant c$ on $\mathbb{R}_+$,

[10] It can be shown that $1/(nx_{\mathrm{d}})$ is the Lagrangian multiplier associated with the constraint in [3.5].

[11] Extending canonical arguments as in Myerson (1993, Theorem 2), it can be shown that G^* has a compact and convex support from 0. This allows us to directly apply DM's results.

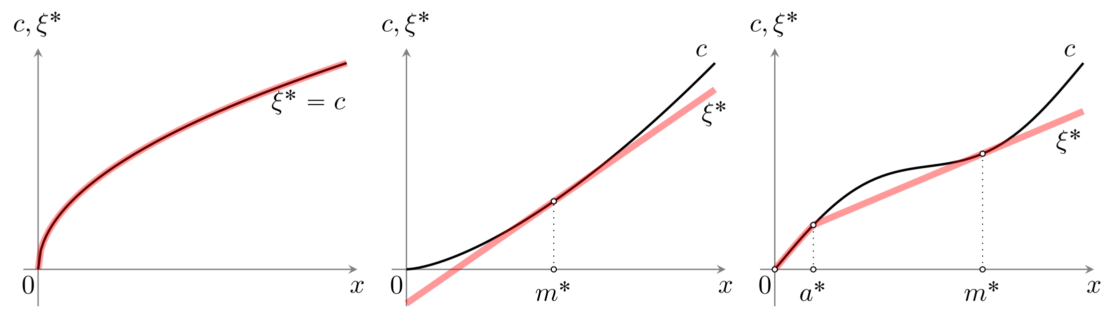

Figure 1: Cost functions (black, solid) and virtual cost functions (red, translucent).

(ii) $\operatorname{supp}(F^*) \subset \{x : \xi^*(x) = c(x)\}$, and (iii) $\int c \, \mathrm{d}F^* = \int \xi^* \, \mathrm{d}F^* = \int \xi^* \, \mathrm{d}G^*$. In addition, ξ^* is a solution to the following dual problem of [4.2]:

$$\text{[4.3]} \qquad \max_{\xi} \int \xi \, \mathrm{d}G^* \qquad \text{s.t.} \quad \xi \in \mathbb{R}^{\mathbb{R}_+} \text{ concave and } \xi \leqslant c \,.$$

In what follows, we refer to ξ^* as the *virtual cost* function; the reason behind this naming will be clarified in the following subsection. Roughly, ξ^* is the highest concave function that stays below c and satisfies $\int \xi^* \, \mathrm{d}F^* = \int \xi^* \, \mathrm{d}G^*$; this latter requirement allows us to select one among potentially many "highest" concave functions below c. As shown in Figure 1, ξ^* coincides with c if, and only if, c is concave (the left panel). If c is convex then ξ^* is an affine function tangent to c at some point m^* (the middle panel). If c is initially concave and then convex, then there exist a^* and m^* such that $\xi^* = c$ below a^* and ξ^* is affine, and tangent to c at m^*, above a^* (the right panel).[12]

4.2. Virtual Contest

For our subsequent results, it is useful to define the following *virtual contest* game.

- Each agent (contestant) now has an adversary.
- Each agent i chooses a distribution G_i over *output* (and not effort).
- Agent i's adversary chooses a *concave* cost function $\xi_i \leqslant c$ to maximise $\int \xi_i \, \mathrm{d}G_i$.

A symmetric equilibrium of the *virtual contest* is a pair (G^*, ξ^*) such that given ξ^*, $G_i = G^*$ solves agent i's problem

$$\text{[4.4]} \qquad \max_{G_i} \left[\int \Phi(G^*(y); \mathbf{v}) \, \mathrm{d}G_i(y) - \int \xi^*(y) \, \mathrm{d}G_i(y) \right],$$

[12] More generally, if c has finitely many inflexion points, then ξ^* consists of finitely many alternating affine and strictly concave segments. The affine segments are tangent to c, while $\xi^* = c$ along the strictly concave segments.

while, given G^*, agent i's adversary's choice of $\xi_i = \xi^*$ solves

$$\max_{\xi_i \in \mathbb{R}^{\mathbb{R}_+}} \int \xi_i(y)\, \mathrm{d}G^*(y) \qquad \text{s.t.} \quad \xi_i \leqslant c,\ \xi_i \text{ concave.} \tag{4.5}$$

The next result shows that there is an isomorphism between our original contest with strategic risk taking and the above virtual contest.

Theorem 1. *Consider the contest with prize schedule* $\mathbf{v} \in \mathscr{V}$. *The distributions* (F^*, G^*) *constitute a symmetric equilibrium of this contest if, and only if,* (G^*, ξ^*) *is a symmetric equilibrium of the corresponding virtual contest where* ξ^* *solves the dual problem* [4.3] *and satisfies all three properties in Proposition 4.1.*

Although the virtual game involves virtual players (the adversaries), it is still an all-pay contest once the virtual cost functions ξ_i have been chosen by the adversaries. This fact allows us to explicitly characterise G^* (given ξ^*) in the virtual contest, as in Barut and Kovenock (1998).

Proposition 4.2. G^* is an equilibrium in the *virtual all-pay contest* with *virtual cost* ξ^*, ie, $G^*(y) = \Phi^{-1}\left(\min\{\xi^*(y) - \xi^*(0), v_1\}; \mathbf{v}\right)$ for all $y \geqslant 0$.

Proof. Notice that [4.4] is linear in G_i. This implies that $\Phi(G^*(y)) - \xi^*(y)$ should be constant in the support of G^*, that is, for any $y \in \operatorname{supp}(G^*)$,

$$\Phi(G^*(y); \mathbf{v}) - \xi^*(y) = \Phi(G^*(0); \mathbf{v}) - \xi^*(0) = -\xi^*(0) \Leftrightarrow \Phi(G^*(y); \mathbf{v}) = \xi^*(y) - \xi^*(0).$$

The desired result follows because globally $\Phi(G^*(y); \mathbf{v}) = \min\{\xi^*(y) - \xi^*(0), v_1\}$. $\square$

Two features distinguish the virtual contest from the standard all-pay contest. First, as illustrated in the middle panel of Figure 1, $\xi^*(0)$—the virtual cost of zero output—can be negative, *even though* $c(0) = 0$. Thus, in equilibrium, each player enjoys a surplus of $-\xi^*(0)$, which is strictly positive if $\xi^*(0) < 0$; as noted in Proposition 4.2, the equilibrium strategy is still such that $\Phi(G^*(y); \mathbf{v}) = \xi^*(y) - \xi^*(0)$ over $\operatorname{supp}(G^*)$. If $\xi^*(0) = 0$, as in the left and right panels of Figure 1, the usual rent-dissipation result holds, so $G^*(y) = \Phi^{-1}(\min\{\xi^*(x), v_1\}; \mathbf{v})$ (cf, Proposition 4.2).

Second, the virtual cost function is not exogenously given, but determined endogenously (by adversaries) in equilibrium; therefore, in general, it depends on the prize schedule $\mathbf{v}$. This suggests that the dependence of G^* (and F^*) on $\mathbf{v}$ is not restricted to the explicit, or *direct*, effect of prizes on function $\Phi(q; \mathbf{v})$, but is also affected *indirectly* by equilibrium adjustments in ξ^*. This point will become more transparent in the subsequent sections where we discuss comparative statics with respect to $\mathbf{v}$.

Even after completely characterising the equilibrium (G^*, ξ^*) of the virtual contest, our task remains incomplete because our primary interest is in the "real" contest, and the distribution of "real" effort. The next result illustrates how one can recover the equilibrium effort distribution F^* from (G^*, ξ^*).

Proposition 4.3. Consider any interval $[y_1, y_2] \subseteq \operatorname{supp}(G^*)$.

(i) If ξ^* is strictly concave on $[y_1, y_2]$ then $F^*(y) = G^*(y)$ for all $y \in [y_1, y_2]$.

(ii) If $[y_1, y_2]$ is the largest interval over which ξ^* is affine, then $F^*(y_i) = G^*(y_i)$ for $i = 1, 2$, and $\int_{y_1}^{y_2} y \,\mathrm{d}F^*(y) = \int_{y_1}^{y_2} y \,\mathrm{d}G^*(y)$.

Proof. Given that $\operatorname{supp}(G^*)$ is an interval, this is a direct application of Proposition 2 of Dworczak and Martini (2019). □

To understand this result, recall property (iii) in Proposition 4.1, namely, that $\int \xi^* \,\mathrm{d}F^* = \int \xi^* \,\mathrm{d}G^*$. If ξ^* is strictly concave (which is the case when c is strictly concave) then this can hold only when $F^* = G^*$; otherwise, $\int \xi^* \,\mathrm{d}F^* > \int \xi^* \,\mathrm{d}G^*$ because ξ^* is strictly concave and $F^* \in \mathrm{MPC}(G^*)$. Now suppose ξ^* is affine, which arises when c is convex. In this case, $\int \xi^* \,\mathrm{d}F^* = \int \xi^* \,\mathrm{d}G^*$ holds as long as F^* has the same mean as G^* and, therefore, for any $F^* \in \mathrm{MPC}(G^*)$. Proposition 4.3 argues that these properties hold locally over any largest interval on which ξ^* is strictly concave or affine.

The results for concave costs follow as a corollary of Propositions 4.1, 4.2 and 4.3.

Corollary 4.1. If c is concave then there is a symmetric equilibrium without risk taking (ie, $F^* = G^*$), where F^* is determined by [**3.2**]. If c is strictly concave then this is the unique symmetric equilibrium. In any symmetric equilibrium, the virtual cost is $\xi^* = c$ and more prize inequality raises the equilibrium expected effort.

5. Contest Design under Convex Effort Costs

This section considers again the case where c is strictly convex, for which we derived the equilibrium in Section 3 leveraging the result from Myerson (1993). In this case, it is immediate that the equilibrium virtual cost ξ^* is linear and touches c only once: Any concave function uniformly below c can meet c at most once. The "highest" such concave functions must be affine and tangent to c at a single point.

Let $m^* (> 0)$ denote the unique point at which ξ^* equals c. Given m^*, the results in Section 4 imply the following: (i) m^* represents the deterministic effort level all players choose, that is, the equilibrium effort distribution F^* is degenerate at m^*; (ii) each player's expected payoff is equal to $-\xi^*(0) = c'(m^*)m^* - c(m^*) > 0$; (iii) the equilibrium output distribution G^* is such that $\Phi(G^*(y); \mathbf{v}) = \min\{c'(m^*)y, v_1\}$.

To determine the equilibrium value of m^*, we define, for each $m \in \mathbb{R}_+$, a function $\xi(\cdot; m)$ as

$$\xi(y; m) = c(m) + c'(m)(y - m),$$

and define a distribution $G(\cdot; m, \mathbf{v})$ by the identity

$$[\mathbf{5.1}] \qquad \Phi(G(y; m, \mathbf{v}); \mathbf{v}) = \min\{\xi(y; m) - \xi(0; m), v_1\} = \min\{c'(m)y, v_1\}.$$

In other words, $\xi(\cdot; m)$ denotes an affine function that is tangent to c at m, while $G(\cdot; m, \mathbf{v})$ denotes the equilibrium distribution in a virtual contest with virtual cost $\xi(y; m)$ (ie, the equilibrium distribution in a standard all-pay contest under cost $\tilde{\xi}(y) = c'(m)y$).

An equilibrium value of m is such that the output distribution $G(\cdot; m, \mathbf{v})$ is a mean-preserving spread of the corresponding effort distribution. In the current case where c is convex, the latter distribution is degenerate at m. It then follows that the mean-preserving spread condition reduces to the mean condition, that is,

$$m = \int y \, \mathrm{d}G(y; m, \mathbf{v}).$$

The following result identifies the unique (equilibrium) value of m that satisfies this condition.

Proposition 5.1. If c is convex then for any $\mathbf{v} \in \mathscr{V}$, there exists a unique symmetric equilibrium in which the players choose a deterministic effort $x_{\mathrm{d}} (> 0)$ such that $x_{\mathrm{d}} c'(x_{\mathrm{d}}) = 1/n$.

Proof. The right-hand side of the above mean condition can be rewritten as

$$\int y \, \mathrm{d}G(y; m, \mathbf{v}) = \int \frac{\Phi(G(y; m, \mathbf{v}); \mathbf{v})}{c'(m)} \, \mathrm{d}G(y; m, \mathbf{v}) = \frac{1}{nc'(m)},$$

where the first equality is due to the definition of G, and the second one is due to a property of Φ, namely, $\int_0^1 \Phi(q; \mathbf{v}) dq = 1/n$.[13] This implies that the above equilibrium mean condition reduces to $mc'(m) = 1/n$. The desired result is then immediate from the fact that when c is convex, $mc'(m)$ is monotone increasing in m. $\square$

We note that, although all contests $\mathbf{v} \in \mathscr{V}$ induce the same deterministic effort (and mean output) x_{d}, different contests produce different output distributions. In particular, the following result follows immediately once the arguments of FNS are combined with the fact that the equilibrium virtual cost function ξ^* is independent of $\mathbf{v} \in \mathscr{V}$.

Corollary 5.1. If c is convex and $\mathbf{w} \in \mathscr{V}$ is more unequal than $\mathbf{v} \in \mathscr{V}$, then the equilibrium output distribution for $\mathbf{w}$ dominates that for $\mathbf{v}$ in the convex order.

[13] Since G is the symmetric equilibrium equilibrium in a virtual contest, each player's expected benefit is $1/n$.

6. Concave-Convex Effort Costs

This section considers the case where c is initially concave and then convex (equivalently, the marginal effort cost c' is U-shaped). Note that, although convex costs are typically assumed in applied work for their technical tractability, concave-convex costs are a textbook example and, in fact, a common justification for (eventually) convex costs.

6.1. Preliminaries

For a clean analysis, we assume that the marginal cost c' is *strictly* quasi-convex (ie, c is first strictly concave and then strictly convex). Let x^ι denote the unique inflexion point, and $\hat{x}$ the point such that $c'(\hat{x})\hat{x} = c(\hat{x})$. Note that, by the structure of c, $x^\iota < \hat{x}$; see Figure 2.

If x^ι is sufficiently large then only the initial concave region is relevant. Then, the equilibrium is just as in the standard all-pay contest without risk taking. By contrast, if $\hat{x}$ is sufficiently small then the equilibrium is fully determined by the convex region of c. Then, the analysis in Section 5 applies effectively unchanged. To focus on novel effects of concave-convex costs, we maintain the following assumption through this section.

Assumption 1. $c(x^\iota) < v_1$ and $c(\hat{x}) > 1/n$.

Indeed, if $c(x^\iota) \geqslant v_1$ then an equilibrium with no risk taking—in which the players play as in the standard all-pay context—exists, while if $c(\hat{x}) \leqslant 1/n$ then there exists a deterministic-effort equilibrium, just as in Section 5. Therefore, Assumption 1 implies that an equilibrium necessarily involves both risk taking over output and randomisation over effort.

6.2. Equilibrium Characterization

Following the general characterization in Section 4, the equilibrium virtual cost function ξ^* takes the form depicted in Figure 2: There exists $m^* \in (x^\iota, \hat{x})$ such that

$$\xi^*(y) = \begin{cases} c(y) & \text{if } y \leqslant a(m^*) \\ c'(m^*)(y - m^*) + c(m^*) & \text{if } y > a(m^*), \end{cases}$$

where $a(m)$ denotes the value below x^ι such that

$$c'(m) = \frac{c(m) - c(a(m))}{m - a(m)} \text{ for each } m \in [x^\iota, \hat{x}].$$

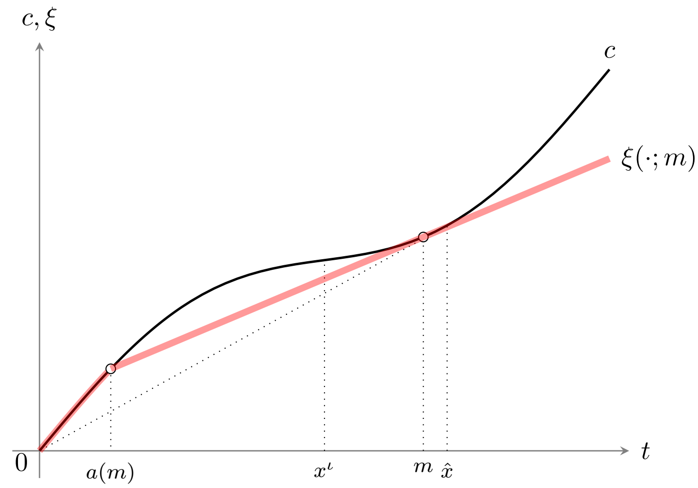

Figure 2: Concave-convex cost function (black, solid) and the corresponding virtual cost function (red, translucent).

In other words, ξ^* coincides with c for $y \leqslant a(m^*)$ and then is affine and tangent to c at $m^* \in (x^\iota, \hat{x})$. Given ξ^* (and $\xi^*(0) = 0$), the equilibrium output distribution G^* is given by

$$G^*(y) = \begin{cases} \Phi^{-1}(\xi^*(y); \mathbf{v}) & \text{if } \xi^*(y) < v_1 \\ 1 & \text{if } \xi^*(y) \geqslant v_1. \end{cases}$$

Finally, the equilibrium effort distribution F^* coincides with G^* below $a(m^*)$ and assigns the remaining probability mass $1 - G^*(a(m^*))$ to m^*.

As in Section 5, equilibrium characterisation boils down to identifying the value of m^*. To that end, for each $m \in (x^\iota, \hat{x})$, we define $\xi(\cdot; m)$ as

$$\xi(y; m) = \begin{cases} c(y) & \text{if } y \leqslant a(m) \\ c'(m)(y - m) + c(m) & \text{if } y > a(m). \end{cases}$$

In addition, we define a distribution $G(\cdot; m)$ by the identity

[**6.1**] $$\Phi(G(y; m); \mathbf{v}) = \min\{\xi(y; m), v_1\}.$$

The equilibrium m^* is the value of m such that

$$\int_{y \geqslant a(m)} (y - m)\, \mathrm{d}G(y; m) = 0.$$

To understand this condition, recall that $F^* \in \text{MPC}(G^*)$. Given the structure of F^* (or, the way F^* is constructed given G^* and ξ^*), the required relationship holds as long as they have the same mean, which, again due to the structure of F^*, reduces to

$$\int y\,\mathrm{d}G^*(y) - \int x\,\mathrm{d}F^*(x) = \int_{y \geqslant a(m)} (y-m)\,\mathrm{d}G^*(y) = 0.$$

Utilizing this condition, we obtain the following result.

Proposition 6.1. If c is strictly concave-convex and satisfies Assumption 1 then there exists a unique equilibrium in which for some $m^* \in (x^\iota, \hat{x})$, the equilibrium effort distribution F^* is continuously and strictly increasing until $a(m^*)$ and then assigns the remaining probability to m^*.

Proof. We provide only a proof sketch here, relegating a complete proof to Appendix A. We define a function $H : [x^\iota, \hat{x}] \to \mathbb{R}$ as

$$H(m) := \int_{y \geqslant a(m)} (y-m)\,\mathrm{d}G(y;m).$$

We first show that under Assumption 1, $H(x^\iota) > 0$, while $H(\hat{x}) < 0$; since H is continuous, this guarantees equilibrium existence. Then, we show that $H'(m) < 0$ whenever $H(m) = 0$; this is weaker than $H' < 0$ everywhere (as in the case of convex costs), but still sufficient for equilibrium uniqueness. □

6.3. Effects of Increasing Prize Inequality

We now analyze how the equilibrium expected effort (and output) $\mathbf{E}[X] = \mathbf{E}[Y]$ depends on the prize allocation $\mathbf{v} \in \mathscr{V}$. Specifically, we consider the effect of $\mathbf{v}$ becoming more unequal. To make the dependence of the equilibrium on $\mathbf{v}$ explicit, we add it as an argument for each relevant object. For example, $X^*(\mathbf{v})$ and $Y^*(\mathbf{v})$ denote the equilibrium random variables under $\mathbf{v} \in \mathscr{V}$.

If the equilibrium virtual cost ξ^* were independent of $\mathbf{v}$ then the effects of more prize inequality would be straightforward from Theorem 0: Since ξ^* is concave, the equilibrium expected effort $\mathbf{E}[X^*(\mathbf{v})]$ necessarily rises as $\mathbf{v}$ becomes more unequal. Therefore, the WTA contest would maximise the expected effort, while the PTB contest would minimise it.

However, ξ^* does depend on $\mathbf{v}$, which makes the relevant comparative statics non-trivial. In the current concave-convex case, nevertheless, the effects through ξ^* are negligible, as formally reported in the following result.

Lemma 6.1. Suppose c is strictly concave-convex and satisfies Assumption 1. Let $Y^*(m, \mathbf{v})$ denote the equilibrium output random variable given virtual cost $\xi(\cdot; m)$. Then, $\mathrm{d}\,\mathbf{E}[Y^*(m^*(\mathbf{v}), \mathbf{v})]/\,\mathrm{d}m = 0$.

Proof. From the identity $\Phi(G(y;m,\mathbf{v});\mathbf{v}) = \min\{\xi(y;m),v_1\}$ and the fact that $\xi(y;m) = c(y)$ if $y < a(m)$ and $\xi(y;m) = c'(m)(y-m)+c(m)$ if $y > a(m)$, one can show that[14]

$$G_m(y;m,\mathbf{v}) = \begin{cases} 0 & \text{if } y < a(m) \\ \frac{c''(m)}{c'(m)}(y-m)g(y;m,\mathbf{v}) & \text{if } y \in [a(m),\overline{y}(m)]. \end{cases}$$

Meanwhile, applying integration by parts and direct differentiation, we obtain

$$\mathbf{E}[Y^*(m,\mathbf{v})] = \int (1-G(y;m,\mathbf{v}))\,\mathrm{d}y \Rightarrow \frac{\mathrm{d}\,\mathbf{E}[Y^*(m,\mathbf{v})]}{\mathrm{d}m} = -\int G_m(y;m,\mathbf{v})\,\mathrm{d}y.$$

Combining the above two results, we arrive at

$$\frac{\mathrm{d}\,\mathbf{E}[Y^*(m,\mathbf{v})]}{\mathrm{d}m} = -\int G_m(y;m,\mathbf{v})\,\mathrm{d}y = -\frac{c''(m)}{c'(m)}H(m).$$

The desired result then follows because $H(m^*) = 0$. □

Thus, Lemma 6.1 establishes that in this case the *indirect* effect of changes in $\mathbf{v}$ on the equilibrium expected output (and effort) is zero. It is important to recognize that Lemma 6.1 is *not* a general result, nor is it a version of the envelope theorem. Among other things, this result relies heavily on the structure of c (and Assumption 1). For example, if c is convex then, as shown in the proof of Proposition 5.1, $\mathbf{E}[Y^*(m,\mathbf{v})] = 1/(nc'(m))$, which is monotone decreasing in m.

Given Lemma 6.1, the following result is straightforward.

Proposition 6.2. If c is strictly concave-convex and satisfies Assumption 1 then the equilibrium expected effort $\mathbf{E}[Y^*(\mathbf{v})]$ increases as $\mathbf{v}$ becomes unequal.

Proposition 6.2 implies the Main Theorem for concave-convex costs. Note that the result is obtained without knowing how the virtual cost $\xi(x;m^*(\mathbf{v}))$ changes with $\mathbf{v}$; in other words, to prove Proposition 6.2 we do not need to know how the equilibrium location of the mass point, $m^*(\mathbf{v})$, is affected by changes in $\mathbf{v}$. Lemma B.1 in Appendix B establishes this comparative static for a restricted set of PD transfers.

7. Convex-Concave Effort Costs

This section considers the case where—opposite to the previous section—c is initially convex and then concave (equivalently, the marginal cost c' is single-peaked).

[14] See the proof of Proposition 6.1 in Appendix A for a comprehensive argument.

7.1. Preliminaries

Similar to Section 6, we assume that the marginal cost c' is *strictly* quasi-concave. We again use x^ι to denote the unique inflection point. If x^ι is sufficiently large then the analysis in Section 5 applies effectively unchanged. However, no matter how small x^ι is, there does not exist an equilibrium without risk taking, because the initial convex region of c can never be ignored. For the current convex-concave case, we proceed without making any assumption on x^ι.

7.2. Equilibrium Characterization

The general characterization in Section 4 can be applied to the current case, just as in Section 6.2. The only difference is the structure of the virtual cost function ξ^*; see Figure 3.

To be formal, for each $m \leqslant x^\iota$, let $b(m)$ denote the point at which the tangent line to c at $(m, c(m))$ meets c in the concave region. If the tangent line uniformly stays below c (which happens, for example, if $m = 0$ and $c'(m) = 0$) then we let $b(m) = \infty$. For each $m \leqslant x^\iota$, we define $\xi(\cdot; m)$ as

$$\xi(y; m) = \begin{cases} c'(m)(y - m) + c(m) & \text{if } y < b(m) \\ c(y) & \text{if } y \geqslant b(m). \end{cases}$$

and $G(\cdot; m)$ as

$$\Phi(G(y; m); \mathbf{v}) = \min\{\xi(y; m) - \xi(0; m), v_1\} = \min\{c'(m)y, v_1\}.$$

Then, the equilibrium m^*—which yields the equilibrium virtual cost $\xi^*(y) = \xi(y; m^*)$—is the value of m such that

$$\int_0^{b(m)} (y - m)\, \mathrm{d}G(y; m) = 0.$$

The existence of equilibrium can be established just as in Section 6.

Proposition 7.1. If c is strictly convex-concave then there exists an equilibrium in which for some $m^* < x^\iota$, the equilibrium effort distribution F^* assigns positive probability to m^* and then continuously increases afters $b(m^*)$.

Proof. Define a function $H(m)$ as follows:

$$H(m) := \int_0^{b(m)} (y - m)\, \mathrm{d}G(y; m).$$

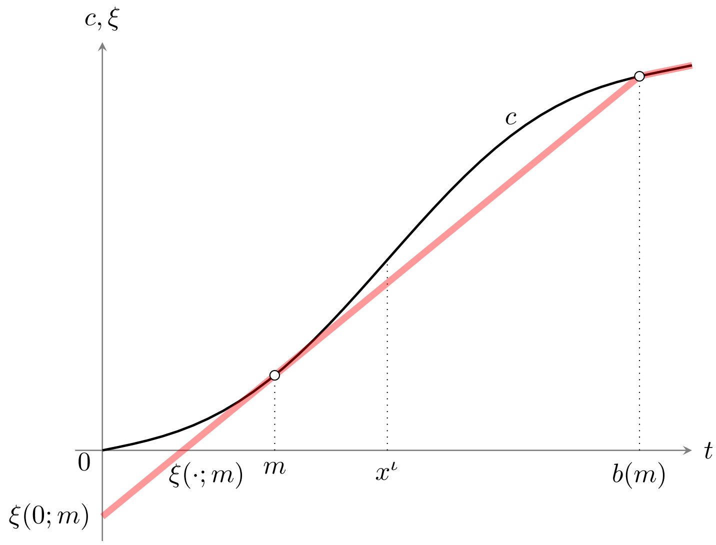

Figure 3: Convex-concave cost function (black, solid) and the corresponding virtual cost function (red, translucent).

If $m = x^\iota$ then $b(m) = x^\iota$, so $H(x^\iota) = \int_0^{x^\iota} (y - x^\iota) \, \mathrm{d}G(y; x^\iota) < 0$. Therefore, it suffices to show that $H(m) > 0$ for m sufficiently small. If $b(0)$ is finite then $H(0) = \int_0^{b(0)} y \, \mathrm{d}G(y; 0) > 0$; if $b(0) = \infty$ (e.g., when $c'(0) = 0$) then pick ε sufficiently small that $\varepsilon c'(\varepsilon) < 1/n$ and $c'(\varepsilon) b(\varepsilon) > v_1$. Then,

$$H(\varepsilon) = \int (y - \varepsilon) \, \mathrm{d}G(y; \varepsilon) = \int \frac{\Phi(G(y; \varepsilon); \mathbf{v})}{c'(\varepsilon)} \, \mathrm{d}G(y; \varepsilon) - \varepsilon = \frac{1}{nc'(\varepsilon)} - \varepsilon > 0.$$

$\square$

Unlike in the previous cases, here equilibrium uniqueness does not hold in general. In Appendix C, we provide a specific example in which there are multiple (symmetric) equilibria.[15] In what follows, we focus on the equilibrium with the smallest possible value of m^* (ie, the equilibrium corresponding to the first value of m such that $H(m) = 0$). As shown in Appendix C, this equilibrium maximises the expected effort among all equilibria, and we refer to it as the equilibrium of the convex-concave case in the next section. Importantly, it follows from Lemma C.1 in Appendix C that the equilibrium is unique in the WTA contest; therefore, if we can show that expected effort increases with prize inequality in the equilibrium with the lowest m^*, it will imply the Main Theorem in this case.

[15] In Appendix C, we also provide sufficient conditions under which equilibrium uniqueness holds, one on prize allocation $\mathbf{v}$ and the other on the cost function c.

7.3. Effects of Increasing Prize Inequality

Now we analyze how the expected effort $\mathbf{E}[Y^*(\mathbf{v})]$ responds to prize inequality. The result is obvious if $nm^*v_1 < b(m^*)$: In this case, the equilibrium effort is deterministic, that is, F^* is degenerate at $m^* = x_{\mathrm{d}}$. Then, as in the convex case, a small change of $\mathbf{v}$ has no impact on the expected effort. From now on, we restrict attention to the case where $nm^*v_1 \geqslant b(m^*)$ (so F^* is not degenerate).[16]

In the previous concave-convex case, we obtain a general comparative statics result, namely, that the expected effort $\mathbf{E}[Y^*(\mathbf{v})]$ always rises as $\mathbf{v}$ becomes more unequal. It holds because the change of m^* (or ξ^*) has a negligible impact on the expected effort, and so the result by FNS with concave costs applies unchanged. This approach no longer works in the current convex-concave case. Following the same steps as in the proof of Lemma 6.1, one can show that

$$\frac{\mathrm{d}\,\mathbf{E}[Y^*(m,\mathbf{v})]}{\mathrm{d}m} = -\int G_m(y;m,\mathbf{v})\,\mathrm{d}y = -\frac{c''(m)}{c'(m)}\int_0^{b(m)} y\,\mathrm{d}G(y;m,\mathbf{v}) < 0.$$

In other words, unlike in the concave-convex case, the effect through the change of m^* is not negligible. This difference arises because $\xi(0;m) < 0$, that is, the players earn positive surplus in the current case (as in the convex case).

Recall that the direct effect of increasing prize inequality always raises $\mathbf{E}[Y^*(\mathbf{v})]$. If m^* falls then the indirect effect works in the same direction (because $\mathrm{d}\,\mathbf{E}[Y^*(m,\mathbf{v})]/ddm < 0$ as shown above). This implies the following (intermediate) result.

Proposition 7.2. In the convex-concave case, as $\mathbf{v}$ becomes more unequal, if m^* falls then $\mathbf{E}[Y^*(\mathbf{v})]$ rises.

Lemma 7.2 would immediately imply our main result—that WTA maximises the expected effort—if m^* always fell with increasing prize inequality. In fact, it is sufficient to establish that m^* decreases with prize inequality for a *subset* of reverse PD transfers that eventually lead to the WTA prize schedule. One such subset is "bottom-reducing" transfers defined as follows.

Definition 7.3. Fix $\mathbf{v} \in \mathscr{V}$, and let k denote the last rank such that $v_k > 0$ (ie, the largest k such that $v_{k+1} = 0$). A *bottom-reducing transfer* is a reverse PD transfer that reduces v_k by $\delta \in (0, v_k)$ and raises v_j by δ for some $j < k$, while ensuring that the resulting prize vector belongs to $\mathscr{V}$.

[16] Because the relevant condition depends on v_1, it is possible that the deterministic-effort equilibrium exists when $\mathbf{v}$ is relatively equitable, but it does not exist once $\mathbf{v}$ becomes sufficiently unequal. This is in contrast to the convex and concave-convex cases where if a deterministic-effort equilibrium exists for some $\mathbf{v} \in \mathscr{V}$ then it does so—with the same F^* but different G^*—for any other $\mathbf{v} \in \mathscr{V}$.

Clearly, starting from any $\mathbf{v} \in \mathscr{V}$, we can reach $\mathbf{v}^{\mathrm{WTA}}$ through a finite number of bottom-reducing transfers. For example, we can sequentially remove the bottom prize (ie, set v_k to 0) and shift it to the top prize.

The following result shows that any bottom-reducing transfer lowers m^*. Combining this with Proposition 7.2 implies that the WTA contest maximizes the expected effort (also) when c is concave-convex.

Lemma 7.4. In the convex-concave case, any bottom-reducing transfer leads to a reduction of m^* (or to a reduction of the lowest m^*, if there are multiple equilibria).

To understand this result, parameterize all relevant objects with δ (the size of reverse PD transfer). Since m^* is the first point at which $H(m, \mathbf{v})$ crosses 0 from above, we have $H_m(m^*, \mathbf{v}) \leqslant 0$. This implies that m^* falls if $H_\delta(m^*, \mathbf{v}) < 0$: In this case, δ and m are *substitutes* for H—which should be equal to 0 in equilibrium. Therefore, m^* moves in the opposite direction of δ. This suggests that the problem reduces to showing that $H_\delta(m^*, \mathbf{v}) < 0$, which is equivalent to showing that $\mathbf{E}[Y^*(m^*, \mathbf{v}) | Y^*(m^*, \mathbf{v}) \leqslant b(m^*)]$ falls in δ. A sufficient condition for this is that the distribution $G(\cdot; m^*, \mathbf{v})$ stochastically decreases conditional on Y being below $b(m^*)$, that is,

$$\frac{\mathrm{d}}{\mathrm{d}\delta}\left[\frac{G(y; m^*, \mathbf{v})}{G(b(m^*); m^*, \mathbf{v})}\right] > 0 \text{ for all } y < b(m^*).$$

This latter condition is equivalent to

$$\frac{G_\delta(y; m^*, \mathbf{v})}{G(y; m^*, \mathbf{v})} > \frac{G_\delta(b(m^*); m^*, \mathbf{v})}{G(b(m^*); m^*, \mathbf{v})} \text{ for all } y < b(m^*).$$

As formally shown in the proof of Lemma 7.4 in Appendix A, this condition holds for all bottom-reducing transfers.

8. Conclusions

We conclude by discussing broader implications of our results for real-world contest design, and a few potential extensions.

Our results are particularly relevant for contest environments where agents are engaged in complex and creative tasks with uncertain outcomes, such as research and innovation contests, architectural design contests, sales contests, or competition for promotion or bonuses in suitable types of organisations. The Main Theorem suggests that a principal interested in maximising expected effort or output would benefit from using more unequal prizes; in particular, the winner-take-all contest maximises the expected total output. Importantly, what allows us to make these broad claims is the unexpected *robustness* of the Main Theorem. Unlike the existing work on contest design that relies substantially on assumptions about the shape of the agents' cost function (and cannot say anything conclusive for costs that are neither globally convex nor concave), our approach suggests that in settings with flexible risk taking this shape is largely irrelevant.

In many cases (eg, when costs are convex), a risk-averse principal who cares about aggregate, or average, output, will also face a trade-off between risk and aggregate efficiency and may prefer to use prize sharing to reduce the variance of effort. For example, for a public research funding agency whose main mission is to support basic research and grow a wide research ecosystem (such as the NSF), it would make sense to fund many projects. The same applies to private foundations focusing on broad agendas, such as the Russel Sage Foundation or the Bill and Melinda Gates Foundation. A similar trade-off is faced by managers in organisations where stakeholders expect stable revenue streams.

A natural extension of our approach is to consider agents with private heterogeneous abilities. In addition to the usual contest design problem, an important application of such a setting is selection contests where the principal's objective is to reward (eg, promote) more able agents. Our techniques allow for a generalisation of Fang and Noe (2022) to continuous distributions of prior abilities. Another application we can generalise is to political competition, similar to Myerson (1993) where we can endogenise politicians' aggregate investments, ie, the "budgets" that politicians have to cultivate minorities. The introduction of endogenous risk taking can also help us contribute to better understanding the moral hazard problem, especially in the context of innovation contests. For instance, in the model of Che and Gale (2003), agents compete in a contest by first making costly investments to determine (private) output and then by participating in a mechanism chosen by the principal. A central assumption in Che and Gale (2003) is that the private research investments completely determine research output. A natural extension of our approach would be to consider such research contests, but where the agents can choose effort *as well as strategic risk*. This clearly changes the incentives for effort and may provide a more robust comparison of different mechanisms and research contest formats.

References

Baker, George P., Michael C. Jensen, and Kevin J. Murphy (1988). "Compensation and incentives: Practice vs. theory." In: *The Journal of Finance* 43.3, pp. 593–616.

Barut, Yasar and Dan Kovenock (1998). "The symmetric multiple prize all-pay auction with complete information." In: *European Journal of Political Economy* 14.4, pp. 627–644.

Bognanno, Michael L. (2001). "Corporate tournaments." In: *Journal of Labor Economics* 19.2, pp. 290–315.

Bretz Jr., Robert D., George T. Milkovich, and Walter Read (1992). "The current state of performance appraisal research and practice: Concerns, directions, and implications." In: *Journal of Management* 18.2, pp. 321–352.

Che, Yeon-Koo and Ian Gale (2003). "Optimal Design of Research Contests." In: *American Economic Review* 93.3, pp. 646–671.

Drugov, Mikhail and Dmitry Ryvkin (2020). "Tournament rewards and heavy tails." In: *Journal of Economic Theory* 190, pp. 105–116.

Dworczak, Piotr and Giorgio Martini (2019). "The simple economics of optimal persuasion." In: *Journal of Political Economy* 127.5, pp. 1993–2048.

Fang, Dawei and Thomas Noe (2022). "Less competition, more meritocracy?" In: *Journal of Labor Economics* 40.3, pp. 669–701.

Fang, Dawei, Thomas Noe, and Philipp Strack (2020). "Turning up the heat: The discouraging effect of competition in contests." In: *Journal of Political Economy* 128.5, pp. 1940–1975.

Gilpatric, Scott M. (2009). "Risk taking in contests and the role of carrots and sticks." In: *Economic Inquiry* 47.2, pp. 266–277.

Hillman, Arye L. and John G. Riley (1989). "Politically contestable rents and transfers." In: *Economics & Politics* 1.1, pp. 17–39.

Hvide, Hans K. (2002). "Tournament rewards and risk taking." In: *Journal of Labor Economics* 20.4, pp. 877–898.

Ke, Changxia, Greg Kubitz, Yang Liu, and Lionel Page (2021). "How tournament incentives shape risk-taking decisions." In: *Working Paper*. https://www.gskubitz.com/uploads/6/2/6/3/62634961/risk_taking_in_rank_order_contest_paper_ii__1_.pdf.

Lazear, Edward P and Sherwin Rosen (1981). "Rank-order tournaments as optimum labor contracts." In: *Journal of political Economy* 89.5, pp. 841–864.

Lim, Noah, Michael J. Ahearne, and Sung H. Ham (2009). "Designing sales contests: Does the prize structure matter?" In: *Journal of Marketing Research* 46.3, pp. 356–371.

Marshall, Albert W., Ingram Olkin, and Barry C. Arnold (2011). *Inequalities: Theory of majorization and its applications*. Springer.

Moldovanu, Benny and Aner Sela (2001). "The optimal allocation of prizes in contests." In: *American Economic Review* 91.3, pp. 542–558.

Myerson, Roger B. (1993). "Incentives to cultivate favored minorities under alternative electoral systems." In: *American Political Science Review* 87.4, pp. 856–869.

Ray, Debraj and Arthur Robson (2012). "Status, Intertemporal Choice, and Risk-Taking." In: *Econometrica* 80.4, pp. 1505–1531.

Shaked, Moshe and J. George Shanthikumar (2007). *Stochastic Orders*. New York, NY: Springer.

Siegel, Ron (2009). "All-pay contests." In: *Econometrica* 77.1, pp. 71–92.

Terwiesch, Christian and Karl T. Ulrich (2009). *Innovation tournaments: Creating and selecting exceptional opportunities*. Harvard Business Press.

Vojnović, Milan (2015). *Contest Theory: Incentive Mechanisms and Ranking Methods*. Cambridge University Press. ISBN: 9781316472903.

A. Proofs

Proof of Proposition 6.1. Define a function $H : [x^\iota, \hat{x}] \to \mathbb{R}$ as

$$H(m) := \int_{y \geqslant a(m)} (y - m) dG(y; m).$$

Note that $\overline{y}(m) := \xi^{-1}(v_1; m)$ is the upper bound of the support of $G(\cdot; m)$.

For equilibrium existence, we prove that under Assumption 1, $H(x^\iota) > 0$, while $H(\hat{x}) < 0$. If $m = x^\iota$ then $a(m) = x^\iota$. Combining this with Assumption 1 yields

$$\xi(x^\iota; m) = c(x^\iota) < v_1 = \xi(\overline{y}(m, v); m) \Rightarrow x^\iota < \overline{y}(m),$$

which ensures $H(x^\iota) > 0$. If $m = \hat{x}$ then $a(m) = 0$, so

$$H(\hat{x}) = \int (y - \hat{x}) dG(y; \hat{x}) = \int \frac{\Phi(G(y; \hat{x}); \mathbf{v})}{c'(\hat{x})} dG(y; \hat{x}) - \hat{x} = \frac{1}{nc'(\hat{x})} - \hat{x}.$$

This is negative because $\hat{x}c'(\hat{x}) = c(\hat{x}) > 1/n$ where the equality is due to the definition of $\hat{x}$ and the inequality is due to Assumption 1.

For uniqueness, we prove that $H'(m) < 0$ whenever $H(m) = 0$. Through integration by parts, H can be rewritten as

$$\begin{aligned} H'(m) &= -(1 - G(a(m); m)) \\ &\quad + (m - a(m))(g(a(m); m)a'(m) + G_m(a(m); m)) - \int_{a(m)}^{\overline{y}(m)} G_m(y; m)dy. \end{aligned}$$

Since $m \geqslant a(m)$ and $a'(m) < 0$, it is sufficient for $H'(m) < 0$ that (i) $G_m(a(m); m) \leqslant 0$ and (ii) $\int_{a(m)}^{\overline{y}(m)} G_m(y; m)dy = 0$.

For both (i) and (ii), recall that for all $y \in [a(m), \overline{y}(m)]$,

$$\Phi(G(y; m); \mathbf{v}) = c(m) + c'(m)(y - m).$$

Differentiating this identity with respect to m and y, we obtain

$$\Phi'(G(y; m); \mathbf{v})G_m(y; m) = c''(m)(y - m) \text{ and } \Phi'(G(y; m); \mathbf{v})g(y; m) = c'(m),$$

leading to

$$G_m(y; m) = \frac{c''(m)}{c'(m)}(y - m)g(y; m).$$

Since $c'(m), c''(m) > 0$ and $a(m) \leqslant m$, (i) $G_m(a(m); m) \leqslant 0$. (ii) also holds whenever $H(m) = 0$, because

$$\int_{a(m)}^{\overline{y}(m)} G_m(x; m)dx = \frac{c''(m)}{c'(m)} \int_{a(m)}^{\overline{y}(m)} (y - m)dG(y; m) = \frac{c''(m)}{c'(m)} H(m).$$

□

Proof of Lemma 7.4. Fix $\mathbf{v} \in \mathscr{V}$, and let $\mathbf{v}^\delta$ denote the prize vector obtained from $\mathbf{v}$ via a bottom-reducing transfer of size δ from k to some $j < k$. We show that m^* is strictly decreasing in δ. Recall that we focus on the smallest value of m^* such that $H(m^*)$, in which case $H_m(m^*, \mathbf{v}) \leqslant 0$ (see the proof of Lemma 7.1). Since $H(m^*(\delta), \mathbf{v}^\delta) = 0$ for any δ, the desired result holds if $H_\delta(m^*(0), v^0) < 0$. We prove this inequality.

We first make a few useful observations. For all $y < b(m^*)$, $\Phi(G(y; m^*, \mathbf{v}); \mathbf{v}) = c'(m^*)y$. Differentiating both sides with δ and evaluating them at $\delta = 0$, we have

[A.1] $$\phi_{jk}(G(y; m^*, \mathbf{v})) + \Phi'(G(y; m^*, \mathbf{v}); \mathbf{v})G_\delta(y; m^*, \mathbf{v}) = 0,$$

where

$$\text{[A.2]} \qquad \phi_{jk}(q) := \binom{n-1}{j-1} q^{n-j}(1-q)^{j-1} - \binom{n-1}{k-1} q^{n-k}(1-q)^{k-1}.$$

Let q_0 be the unique point at which $\phi_{jk}(q_0) = 0$. Then, $\phi_{jk}(q) < 0$ if $q < q_0$, while $\phi_{jk}(q) > 0$ if $q > q_0$. Combined with [A.1], this implies that $G_\delta(y; m^*, \mathbf{v}) > 0$ if $q < q_0$, while $G_\delta(y; m^*, \mathbf{v}) < 0$ if $q > q_0$.

Via integration by parts,

$$H(m, \mathbf{v}) = \int_0^{b(m)} (y-m) dG(y; m, \mathbf{v}) = (b(m) - m)G(b(m); m, \mathbf{v}) - \int_0^{b(m)} G(y; m, \mathbf{v}) dy.$$

Differentiating this with respect to δ and evaluating it at $(m, \delta) = (m^*, 0)$, we obtain

$$H_\delta(m^*, \mathbf{v}) = (b(m^*) - m^*)G_\delta(b(m^*); m^*, \mathbf{v}) - \int_0^{b(m^*)} G_\delta(y; m^*, \mathbf{v}) dy.$$

Combining [A.1] with the fact that $\Phi'(G(y; m^*, \mathbf{v}); \mathbf{v})g(y; m^*, \mathbf{v}) = c'(m^*)$ yields

$$-\int_0^{b(m^*)} G_\delta(y; m^*, \mathbf{v}) dy = \frac{1}{c'(m^*)} \int_0^{G(b(m^*); m^*, \mathbf{v})} \phi_{jk}(q) dq < \frac{1}{c'(m^*)} \int_0^1 \phi_{jk}(q) dq = 0,$$

where the inequality is because $\phi_{jk}(q) > 0$ for $q > q_0$. There are the following two cases to consider: (i) $G_\delta(b(m^*); m^*, \mathbf{v}) \leqslant 0$ and (ii) $G_\delta(b(m^*); m^*, \mathbf{v}) > 0$. The result ($H_\delta(m^*, \mathbf{v}) < 0$) is straightforward in the former case.

Consider the case where $G_\delta(b(m^*); m^*, \mathbf{v}) > 0$, which, by the result above, is equivalent to $G(b(m^*); m^*, \mathbf{v}) < q_0$; we use the properties of bottom-reducing transfers for this part of the proof. Using $H(m^*, \mathbf{v}) = 0$, $H_\delta(m^*, \mathbf{v})$ can be rewritten as

$$\begin{aligned} H_\delta(m^*, \mathbf{v}) &= \frac{G_\delta(b(m^*); m^*, \mathbf{v})}{G(b(m^*); m^*, \mathbf{v})} \int_0^{b(m^*)} G(y; m^*, \mathbf{v}) dy - \int_0^{b(m^*)} G_\delta(y; m^*, \mathbf{v}) dy \\ &= \frac{G_\delta(b(m^*); m^*, \mathbf{v})}{G(b(m^*); m^*, \mathbf{v})} \int_0^{b(m^*)} G_\delta(y; m^*, \mathbf{v}) \left[\frac{G(y; m^*, \mathbf{v})}{G_\delta(y; m^*, \mathbf{v})} - \frac{G(b(m^*); m^*, \mathbf{v})}{G_\delta(b(m^*); m^*, \mathbf{v})} \right] dy. \end{aligned}$$

For $H_\delta(m^*, \mathbf{v}) < 0$, it is sufficient that G/G_δ is *increasing* in y—as it implies that the braketed term is negative for any $y \leqslant b(m^*)$—or equivalently, that $R(q) := -q\Phi'_v(q)/\phi_{jk}(q)$ is increasing in q for $q < q_0$, where we have used [A.1] and set $q = G(y; m^*, \mathbf{v})$. Using their definitions, it can be shown that

$$\text{[A.3]} \qquad q\Phi'(q; \mathbf{v}) = (n-1)q^{n-k}(1-q)^{k-1} \sum_{i=1}^{k} \binom{n-2}{i-1} z^{k-i} \Delta v_i$$

and

$$\phi_{jk}(q;\mathbf{v}) = \binom{n-1}{j-1} q^{n-k}(1-q)^{k-1}(z^{k-j} - z_0^{k-j})$$

where $z = q/(1-q)$ and $z_0 = q_0/(1-q_0)$. Now, $R(q)$ can be written as

[**A.4**] $$R(q) = -\frac{q\Phi'(q;\mathbf{v})}{\phi_{jk}(q)} = \frac{(n-1)\sum_{i=1}^{k}\binom{n-2}{i-1}z^{k-i}\Delta v_i}{\binom{n-1}{j-1}(z_0^{k-j} - z^{k-j})}.$$

Clearly, the numerator is increasing in z and the denominator is decreasing in z; therefore, $R(q)$ is increasing in z as required. □

B. Further Comparative Statics Results for Concave-Convex Costs

Lemma B.1. Suppose c is concave-convex. Consider two prize schedules, $\mathbf{v}$ and $\mathbf{w}$, such that $\mathbf{w}$ is more unequal and is obtained from $\mathbf{v}$ via a transfer $(1, j)$, $j > 1$. Then $m^*(\mathbf{w}) \geqslant m^*(\mathbf{v})$.

Proof. Consider a family of prize schedules $\mathbf{v}^\delta$, where $\mathbf{v}^0 = \mathbf{v}$ and $\mathbf{v}^\delta$ is obtained from v via a transfer $(1, j)$. It is sufficient to show that, for a fixed $m^* = m^*(\mathbf{v})$, $H(m^*, \mathbf{v}^\delta)$ is increasing in δ at $\delta = 0$, i.e., that $H_\delta(m^*, \mathbf{v}^\delta)|_{\delta=0} > 0$.

For brevity, let $a^* = a(m^*)$. Differentiating $H(m, v)$ with respect to δ,

$$H_\delta(m^*, \mathbf{v}^0) = \bar{x}_\delta(m^*, \mathbf{v}) + (m^* - a^*)G_\delta(a^*; m^*, \mathbf{v}) - \bar{x}_\delta(m^*, \mathbf{v}) - \int_{a^*}^{\bar{x}(m^*,\mathbf{v})} G_\delta(x; m^*, \mathbf{v})dx$$

[**B.1**]

$$= (m^* - a^*)G_\delta(a^*; m^*, \mathbf{v}) - \int_{a^*}^{\bar{x}(m^*,\mathbf{v})} G_\delta(x; m^*, \mathbf{v})dx.$$

From $\Phi(G(x; m, \mathbf{v}); \mathbf{v}) = c(m) + c'(m)(x - m)$, we obtain $G_\delta = -\phi_{1j}(G)/\Phi'_v(G)$. Combining this with $\Phi'(G(x; m, \mathbf{v}); \mathbf{v})g(x; m, \mathbf{v}) = c'(m)$, we obtain

$$-\int_{a^*}^{\bar{x}(m^*,\mathbf{v})} G_\delta(x; m^*, \mathbf{v})dx = \int_{a^*}^{\bar{x}(m^*,\mathbf{v})} \frac{\phi_{1j}(G(x; m^*, \mathbf{v}))}{c'(m)} g(x; m^*, \mathbf{v})dx$$
$$= \frac{1}{c'(m)} \int_{G(a^*;m^*,\mathbf{v})}^{1} \phi_{1j}(q)dq > 0.$$

The inequality follows because $\phi_{1j}(q)$ is a single-crossing function of q, first negative, then positive, and integrates to zero on $[0, 1]$. Let q_0 denote the crossing point. In order to sign (B.1), there are two cases to consider: (i) $G_\delta(a^*; m^*, \mathbf{v}) \geqslant 0$ or, equivalently,

$G(a^*; m^*, \mathbf{v}) \leqslant q_0$, in which case the result follows immediately; and (ii) $G_\delta(a^*; m^*, \mathbf{v}) < 0$ or, equivalently, $G(a^*; m^*, \mathbf{v}) > q_0$, in which case the first term in (B.1) is negative and additional steps are needed.

We now consider case (ii), i.e., assume $G_\delta(a^*; m^*, \mathbf{v}) < 0$ and $G(a^*; m^*, \mathbf{v}) > q_0$. $H(m, \mathbf{v})$ can be rewritten as

$$H(m, \mathbf{v}) = \bar{x} - a(m) + (a(m) - m)(1 - G(a(m); m, \mathbf{v})) - \int_{a(m)}^{\bar{x}(m,\mathbf{v})} G(x; m, \mathbf{v}) dx$$
$$= \int_{a(m)}^{\bar{x}(m,\mathbf{v})} [1 - G(x; m, \mathbf{v})] dx - (m - a(m))(1 - G(a(m); m, \mathbf{v})).$$

The condition $H(m^*, \mathbf{v}) = 0$ then gives

$$m^* - a^* = \frac{1}{1 - G(a^*; m^*, \mathbf{v})} \int_{a^*}^{\bar{x}(m^*,\mathbf{v})} [1 - G(x; m^*, \mathbf{v})] dx,$$

which allows us to write [**B.1**] in the form

$$H_\delta(m^*, \mathbf{v}^0) = \frac{G_\delta(a^*; m^*, \mathbf{v})}{1 - G(a^*; m^*, \mathbf{v})} \int_{a^*}^{\bar{x}(m^*,\mathbf{v})} [1 - G(x; m^*, \mathbf{v})] dx - \int_{a^*}^{\bar{x}(m^*,\mathbf{v})} G_\delta(x; m^*, \mathbf{v}) dx$$
$$= \int_{a^*}^{\bar{x}(m^*,\mathbf{v})} [1 - G(x; m^*, \mathbf{v})] \left[\frac{G_\delta(a^*; m^*, \mathbf{v})}{1 - G(a^*; m^*, \mathbf{v})} - \frac{G_\delta(x; m^*, \mathbf{v})}{1 - G(x; m^*, \mathbf{v})} \right] dx.$$

To show that $H_\delta(m^*, \mathbf{v}^0) > 0$ it is, therefore, sufficient to show that $G_\delta(x; m^* v)/[1 - G(x; m^*, \mathbf{v})]$ is decreasing in x for $G(x; m^*, \mathbf{v}) > q_0$ or, equivalently, $R(q) = \phi_{1j}(q)/[\Phi_v'(q)(1 - q)]$ is increasing in q for $q > q_0$.

From $\Phi_v(q) := \sum_{k=1}^n \binom{n-1}{k-1} (1-q)^{k-1} q^{n-k} v_k$,

$$\Phi'(q; \mathbf{v}) = \sum_{i=1}^n \binom{n-1}{i-1} (n-i) q^{n-i-1} (1-q)^{i-1} v_i - \sum_{i=1}^n \binom{n-1}{i-1} (i-1) q^{n-i} (1-q)^{i-2} v_i$$
$$= \sum_{i=1}^n \frac{(n-1)!}{(i-1)!(n-1-i)!} q^{n-i-1} (1-q)^{i-1} v_i - \sum_{i=2}^n \frac{(n-1)!}{(i-2)!(n-i)!} q^{n-i} (1-q)^{i-2} v_i$$
$$= (n-1) \sum_{i=1}^{n-1} \binom{n-2}{i-1} q^{n-i-1} (1-q)^{i-1} v_i - (n-1) \sum_{i=1}^{n-1} \binom{n-2}{i-1} q^{n-i-1} (1-q)^{i-1} v_{i+1}$$

[**B.2**]

$$= (n-1) \sum_{i=1}^{n-1} \binom{n-2}{i-1} q^{n-i-1} (1-q)^{i-1} \Delta v_i,$$

where $\Delta v_i = v_i - v_{i+1} \geqslant 0$.[17] Furthermore, from [A.2] for a transfer $(1, j)$ we have

$$\text{[B.3]} \quad \phi_{1j}(q) = q^{n-1} - \binom{n-1}{j-1} q^{n-j}(1-q)^{j-1} = q^{n-1} - \left(\frac{q_0}{1-q_0}\right)^{j-1} q^{n-j}(1-q)^{j-1},$$

where q_0 is such that $\phi_{1j}(q_0) = 0$. Letting $z = q/(1-q)$ and $z_0 = q_0/(1-q_0)$, obtain

$$\begin{aligned} R(q) = \frac{\phi_{1j}(q)}{\Phi_v'(q)(1-q)} &= \frac{q^{n-1} - z_0^{j-1} q^{n-j}(1-q)^{j-1}}{(n-1)\sum_{i=1}^{n-1}\binom{n-2}{i-1} q^{n-i-1}(1-q)^i \Delta v_i} \\ &= \frac{z^{n-1} - z_0^{j-1} z^{n-j}}{(n-1)\sum_{i=1}^{n-1}\binom{n-2}{i-1} z^{n-i-1}\Delta v_i}. \end{aligned}$$

The derivative of $R(q)$ with respect to z is, up to a positive multiplier,

$$\begin{aligned} &\partial_z R(q) \propto [(n-1)z^{n-2} - (n-j)z_0^{j-1} z^{n-j-1}] \sum_{i=1}^{n-1} \binom{n-2}{i-1} z^{n-i-1}\Delta v_i \\ &- (z^{n-1} - z_0^{j-1} z^{n-j}) \sum_{i=1}^{n-1} \binom{n-2}{i-1} (n-i-1) z^{n-i-2}\Delta v_i \\ &= \sum_{i=1}^{n-1} \binom{n-2}{i-1} z^{n-i-1}\Delta v_i z^{2n-i-j-2}[(n-1)z^{j-1} - (n-j)z_0^{j-1} - (n-i-1)(z^{j-1} - z_0^{j-1})] \\ &= \sum_{i=1}^{n-1} \binom{n-2}{i-1} z^{n-i-1}\Delta v_i z^{2n-i-j-2}[iz^{j-1} + (j-i-1)z_0^{j-1}] \\ &> \sum_{i=1}^{n-1} \binom{n-2}{i-1} z^{n-i-1}\Delta v_i z^{2n-i-j-2}(j-1)z_0^{j-1} > 0, \end{aligned}$$

where the first inequality is due to $z > z_0$. Thus, $R(q)$ is increasing in z for $z > z_0$ and hence it is also increasing in q for $q > q_0$. □

Lemma B.1 describes the equilibrium adjustment in the virtual cost function $\xi^*(x)$ in response to *top-improving* transfers $(1, j)$. It is easy to see that if the mass point shifts up, $m^*(\mathbf{w}) > m^*(\mathbf{v})$, then $F^*(\cdot; \mathbf{w})$ crosses $F^*(\cdot; \mathbf{v})$ once from above. Since we already know from Proposition 6.2 that the expected effort goes up as well, Theorem 4.A.22 from Shaked and Shanthikumar (2007) implies the following result.

Proposition B.2. Suppose c is concave-convex and prize schedules $v, w \in \mathscr{V}$ are such that w is obtained from v via a sequence of top-improving transfers. Then F_w^* dominates F_v^* in the increasing convex order.

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[17] This representation shows that $\Phi_v'(q)$ is proportional to the benefit of a player whose output quantile is q in a contest with $n-1$ players and possibly nonmonotone prizes Δv_i. Therefore, $\Phi_v'(q)$ is positive but not necessarily monotone in q (its monotonicity depends on whether v_i are convex or concave in i).

C. Multiple equilibria for convex-concave costs

The following lemma provides sufficient conditions for the equilibrium uniqueness.

Lemma C.1. The equilibrium is unique in the convex-concave case if either of the following conditions hold:

(a) $\Phi_v(q)$ is convex in q;

(b) $c'(b(m))b(m) \geqslant c'(m)m$ for all $m \in (0, x^I]$.

Proof. It is convenient to define a modified version of function H:

$$\tilde{H}(m,v) = c'(m)\int_0^{b(m)}(x-m)dG(x;m,v).$$

Showing that $\tilde{H}$ is single-crossing from positive to negative in m is, of course, equivalent to showing the same for H. It is also convenient to define $B = G(b(m);m,v)$ and $M = G(m;m,v)$, both functions of m and v.

Recall that $\Phi_v(G) = c'(m)x$ for $x \in [0, \min\{b(m), \bar{x}\}]$; therefore,

$$\tilde{H}(m,v) = \int_0^B \Phi_v(q)dq - c'(m)mB. \tag{C.1}$$

Differentiating with respect to m, obtain

$$\tilde{H}_m(m,v) = [\Phi_v(B) - c'(m)m]\frac{dB}{dm} - [c''(m)m + c'(m)]B.$$

If $b(m) > \bar{x}(m,v)$, we have $B = 1$ and $\tilde{H}_m < 0$ as required. Suppose $b(m) \leqslant \bar{x}(m,v)$, in which case $\Phi_v(B) = c'(m)b(m)$ and

$$\frac{dB}{dm} = \frac{1}{\Phi_v'(B)}[c''(m)b(m) + c'(m)b'(m)]. \tag{C.2}$$

Part (a): From $c(b) - c(m) = c'(m)(b-m)$, we can show that

$$c''(m)b(m) + c'(m)b'(m) = c''(m)m + c'(b(m))b'(m) \leqslant c''(m)m, \tag{C.3}$$

implying $dB/dm \leqslant c''(m)m/\Phi_v'(B)$. Note also that $\Phi_v(M) = c'(m)m$. Thus,

$$\begin{aligned}\tilde{H}_m(m,v) &\leqslant [\Phi_v(B) - \Phi_v(M)]\frac{c''(m)m}{\Phi_v'(B)} - [c''(m)m + c'(m)]B \\ &< \frac{c''(m)m}{\Phi_v'(B)}[\Phi_v(B) - \Phi_v(M) - \Phi_v'(B)B] < 0,\end{aligned}$$

where the last inequality follows from the convexity of $\Phi_v(q)$ in q.

Part (b): The condition in part (b) simply ensures that $dB/dm \leqslant 0$. Indeed, combining (C.2) and (C.3), obtain that the sign of dB/dm is the same as the sign of

$$c''(m)b(m) + c'(m)b'(m) = c''(m)b(m) + \frac{c'(m)c''(m)(b(m)-m)}{c'(b(m)) - c'(m)}$$
$$= \frac{c''(m)[c'(b(m))b(m) - c'(m)m]}{c'(b(m)) - c'(m)} \leqslant 0.$$

The inequality follows from the condition in part (a), and the assumption that $c(x)$ is first convex then concave that ensures the denominator is negative. □

The convexity of $\Phi_v(q)$ imposes a restriction on prize schedules. It is easy to show that $\Phi_v(q)$ is convex provided prize differentials Δv_i are decreasing in i, i.e., the prize schedule v itself is convex. The winner-take-all schedule satisfies this property.

The condition in part (b) of Lemma C.1 essentially states that costs are not too flat beyond the inflexion point. If costs become flat, there may be multiple equilibria: a "high m" equilibrium where the mass point is at a relatively high effort and there is little (or no) effort mixing above x^I, and a "low m" equilibrium where the mass point is at a low effort and agents leverage the flatness of the cost function with mixing over a wider range at high efforts. The existence of multiple equilibrium is demonstrated with thew following example.

Example C.2 (Multiple equilibria)**.** Consider a cost function of the form

$$c(x) = \begin{cases} x^s, & x \in [0, \mu] \\ \mu^s + \alpha(x - \mu), & x \geqslant \mu \end{cases}$$

for some $\mu > 0$, $\alpha > 0$, $s > 1$. This function, illustrated in the left panel of Figure 4, is strictly convex for $x \in [0, \mu]$ and affine for $x \geqslant \mu$.[18] The corresponding $b(m)$ satisfies

$$sm^{s-1}(b - m) = \mu^s + \alpha(b - \mu) - m^s,$$

which gives

$$b(m) = \begin{cases} \frac{(s-1)m^s + \mu^s - \alpha\mu}{sm^{s-1} - \alpha}, & sm^{s-1} > \alpha \\ \infty, & \text{otherwise} \end{cases}$$

Further, we consider the punish-the-bottom prize schedule, $v = (\frac{1}{n-1}, \ldots, \frac{1}{n-1}, 0)$, with $\Phi_v(q) = [1-(1-q)^{n-1}]/(n-1)$. This $\Phi_v(q)$ is strictly concave in q, which, in conjunction

[18] It is not differentiable at $x = \mu$, but this point can never be in the support of F, so it does not affect the analysis.

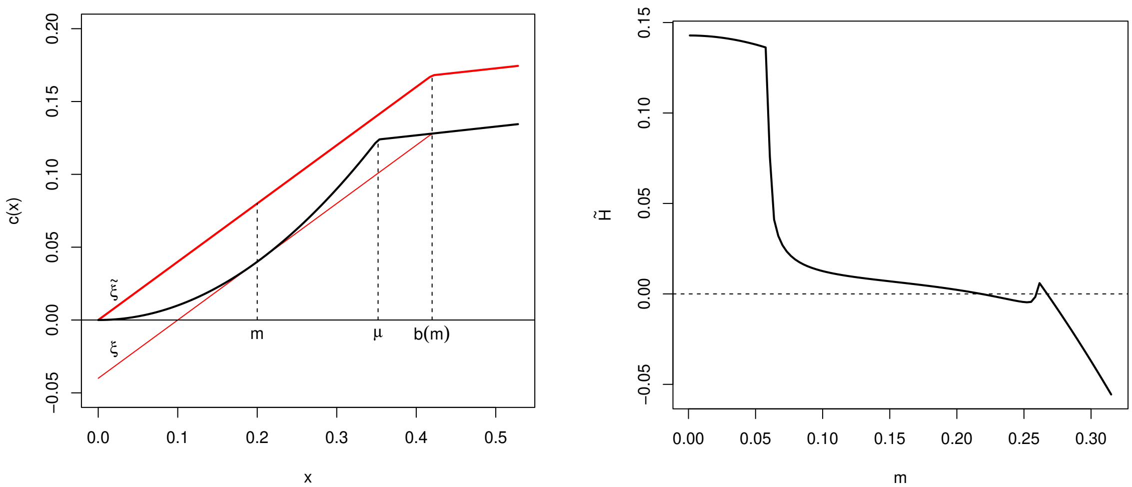

Figure 4: An example with multiple equilibria (Example C.2). Left: The cost function $c(x)$, with the corresponding functions $\xi(x,m)$ and $\tilde{\xi}(x,m)$ illustrated by the thin and thick red lines, respectively. Right: Function $\tilde{H}(m,v)$. Parameters: $\mu = 0.315$, $\alpha = 0.06$, $s = 2$, $n = 7$, the punish-the-bottom prize schedule.

with a low enough α in the cost function, will produce large effects to counteract the sufficient conditions in Lemma C.1.

Let $B(m) = G(b(m); m, v)$. For a given $m \in (0, \mu]$, we have $B = 1$ if $c'(m)b(m) \geqslant v_1 = 1/(n-1)$; that is, $B = 1$ whenever $sm^{s-1} \leqslant \alpha$ or

$$\frac{sm^{s-1}[(s-1)m^s + \mu^s - \alpha\mu]}{sm^{s-1} - \alpha} \geqslant \frac{1}{n-1}.$$

Otherwise, B satisfies the equation

$$\frac{sm^{s-1}[(s-1)m^s + \mu^s - \alpha\mu]}{sm^{s-1} - \alpha} = \frac{1}{n-1}[1 - (1-B)^{n-1}],$$

which can be solved in a closed form.

We use the function $\tilde{H}$ defined in the proof of Lemma C.1. Equation (C.1) gives

$$\begin{aligned}\tilde{H}(m,v) &= \frac{1}{n-1}\int_0^{B(m)} [1-(1-q)^{n-1}]dq - sm^s B(m) \\ &= \frac{1}{n-1}\left[B(m) - \frac{1-(1-B(m))^n}{n}\right] - sm^s B(m).\end{aligned}$$

The resulting function $\tilde{H}(m,v)$ for a specific combination of parameters is shown in the right panel of Figure 4. There are three points where $\tilde{H}(m, v) = 0$.

Given the possibility of the multiplicity of equilibria, it is useful to be able to rank equilibria in a natural way. The following result shows that they are ranked in terms of output and the agents' payoff by the location of the mass point m^*.

Lemma C.3. Multiple equilibria in the convex-concave case, if present, are ranked by m^*. Output decreases with m^* in the FOSD sense, and the agents' payoff increases with m^*.

Proof. It is easy to see that function $\tilde{\xi}(x, m)$ increases in m. Therefore, $G(x; m, v)$ is increasing in m as well, and hence, output is decreasing in m^* in the FOSD sense across multiple equilibria. Furthermore, $-\xi(0, m) = c'(m)m - c(m)$, which gives $-\xi_m(0, m) = c''(m)m > 0$ for $m \in (0, x^I)$, implying that the agents' equilibrium payoff is increasing in m^*. □

Lemma C.3 implies that, generically, the equilibrium with the highest expected effort satisfies $H_m(m^*, v) < 0$.

D. General cost functions

In this section we show how the techniques developed for concave-convex and convex-concave costs with one inflexion points can be extended to establish the Main Theorem for more general cost functions.

Proposition D.1. Suppose c is differentiable, strictly increasing, and has finitely many inflexion point. Then the expected equilibrium effort (and output) is maximised by the winner-take-all contest.

Proof. Consider a $v \in \mathscr{V}$ and assume an equilibrium (G_v^*, F_v^*) exists. Let ξ_v^* denote the corresponding virtual cost function. Since c has finitely many inflexion points, $\xi^*(x)$ consists of finitely many alternating affine and strictly concave segments in the support of G_v^*. Suppose there are $K \geqslant 0$ affine segments, and let $[a_k, b_k]$ denote the corresponding intervals in $\mathrm{supp}(G_v^*)$. In each of the affine segments, let m_k^* denote a point where $\xi^*(x)$ is tangent, and equal, to $c(x)$.[19] We, therefore, consider a parameterized family of local perturbations of ξ_v^* in the form

$$\xi(x; \mathbf{m}) = \begin{cases} c(m_k) + c'(m_k)(x - m_k), & x \in [a_k, b_k] \\ c(x), & \text{otherwise} \end{cases}$$

[19]If $c(x)$ is affine in some intervals, it is possible that $\xi^*(x) = c(x)$ in an interval. In that case, let m_k^* be in the middle of the interval.

where $\mathbf{m} = (m_1, \ldots, m_K)$ is a vector of tangency points, and $\xi_v^*(x) = \xi(x; \mathbf{m}^*)$.[20]

Case 1: $a_1 > 0$, ie, the initial segment of ξ_v^* is strictly concave. In this case $\xi_v^*(0) = 0$, and G_v^* satisfies the identity $\Phi_v(G_v^*(x)) = \min\{\xi_v^*(x), v_1\}$. Let distribution $G(x; \mathbf{m}, v)$ be defined by the identity

$$\Phi_v(G(x; \mathbf{m}, v)) = \min\{\xi(x; \mathbf{m}), v_1\}, \tag{D.1}$$

with $G_v^*(x) = G(x; \mathbf{m}^*, v)$. Let $\bar{x}(\mathbf{m}^*, v)$ denote the upper bound of the support of G_v^*, and write the equilibrium expected effort as

$$x_v = \int x dG_v^*(x) = \bar{x}(\mathbf{m}^*, v) - \int_0^{\bar{x}(\mathbf{m}^*, v)} G_v^*(x) dx.$$

Consider a P-D transfer (i, j) that makes v more unequal by transferring $\delta > 0$ from prize v_j to prize v_i, $i < j$. As before, we can construct a family of prize schedules v^δ obtained from $v = v^0$ via such transfers. It is sufficient to consider the effect of an infinitesimal transfer on x_v, ie, $[dx_{v^\delta}/d\delta]_{\delta=0}$.

We have

$$\left.\frac{dx_{v^\delta}}{d\delta}\right|_{\delta=0} = \frac{d\bar{x}}{d\delta} - \frac{d\bar{x}}{d\delta} G_v^*(\bar{x}) - \int_0^{\bar{x}(\mathbf{m}^*, v)} \frac{d}{d\delta} G_v^*(x) dx \tag{D.2}$$

$$= -\int_0^{\bar{x}(\mathbf{m}^*, v)} \left[\sum_{k=1}^{K} G_{m_k}(x; \mathbf{m}^*, v) \frac{dm_k^*}{d\delta} + G_\delta(x; \mathbf{m}^*, v) \right] dx. \tag{D.3}$$

Here, all derivatives are evaluated at $\delta = 0$ and $\mathbf{m} = \mathbf{m}^*$, ie, at the original equilibrium point. Differentiating [D.1] with respect to x, m_k, and δ, obtain, for $x \in \operatorname{supp}(G_v^*)$:

$$\Phi_v'(G(x; \mathbf{m}^*, v)) g(x; \mathbf{m}^*, v)) = \xi_x(x; \mathbf{m}^*) = \begin{cases} c'(m_k^*), & x \in [a_k, b_k] \\ c'(x), & \text{otherwise} \end{cases} \tag{D.4}$$

$$\Phi_v'(G(x; \mathbf{m}^*, v)) G_{m_k}(x; \mathbf{m}^*, v) = \xi_{m_k}(x; \mathbf{m}^*) = \begin{cases} c''(m_k^*)(x - m_k^*), & x \in [a_k, b_k] \\ 0, & \text{otherwise} \end{cases} \tag{D.5}$$

[20]For $\mathbf{m}$ deviating sufficiently far from $\mathbf{m}^*$, the structure of $\xi(x; \mathbf{m})$ may become very different from that of $\xi_v^*(x)$; in particular, because a_k and b_k shift with m_k, the number of intervals where $\xi(x; \mathbf{m})$ is affine may change. However, for our purposes it is sufficient to consider local perturbations of $\mathbf{m}$ around $\mathbf{m}^*$ such that the structure of $\xi_v^*(x)$ is preserved.

[D.6] $$\phi_{ij}(G(x;\mathbf{m}^*,v)) + \Phi_v'(G(x;\mathbf{m}^*,v))G_\delta(x;\mathbf{m}^*,v) = 0$$

Combining [**D.4**] and [**D.5**], we can write

$$\int_0^{\bar{x}(\mathbf{m}^*,v)} \sum_{k=1}^K G_{m_k}(x;\mathbf{m}^*,v)\frac{\mathrm{d}m_k^*}{\mathrm{d}\delta}\,\mathrm{d}x = \sum_{k=1}^K \frac{c''(m_k^*)}{c'(m_k^*)}\frac{\mathrm{d}m_k^*}{\mathrm{d}\delta}\int_{a_k}^{b_k}(x-m_k^*)\,\mathrm{d}G(x;\mathbf{m}^*,v) = 0.$$

Each term in the last sum is equal to zero. Indeed, if c is affine at m_k^*, we have $c''(m_k^*) = 0$. Otherwise, each m_k^* is chosen optimally such that $\xi(x;\mathbf{m}^*)$ solves the dual problem [**??**], whose first-order conditions, $\left[\frac{\partial}{\partial m_k}\int \xi(x;\mathbf{m})\,\mathrm{d}G_v^*(x)\right]_{\mathbf{m}=\mathbf{m}^*} = 0$, are equivalent to $\int_{a_k}^{b_k}(x-m_k^*)\,\mathrm{d}G(x;\boldsymbol{m}^*,v) = 0$.

Furthermore, combining [**D.4**] and [**D.6**], we can write

$$-\int_0^{\bar{x}(\mathbf{m}^*,v)} G_\delta(x;\mathbf{m}^*,v)\,\mathrm{d}x = \int_0^{\bar{x}(\mathbf{m}^*,v)} \frac{\phi_{ij}(G(x;\mathbf{m}^*,v))}{\xi_x(x;\mathbf{m}^*)}\,\mathrm{d}G(x;\mathbf{m}^*,v)) \geqslant 0,$$

following the same argument as in the proof of Proposition 6.2. Thus, $[dx_{v^\delta}/d\delta]_{\delta=0} \geqslant 0$ as required. The inequality is strict with if $\xi(x;\mathbf{m}^*)$ is nonlinear, ie, whenever effort mixing is involved. The overall result is, therefore, a generalization of Proposition 6.2, and implies the Main Theorem.

Case 2: $a_1 = 0$, ie, the initial segment of ξ_v^* is affine. In this case $\xi_v^*(0) \leqslant 0$ (generically, < 0), and G_v^* satisfies the identity $\Phi_v(G_v^*(x)) = \min\{\tilde{\xi}_v^*(x), v_1\}$, where $\tilde{\xi}_v^*(x) = \xi_v^*(x) - \xi_v^*(0)$. Defining $\xi(x;\mathbf{m})$ as above, and $\tilde{\xi}(x;\mathbf{m}) = \xi(x;\mathbf{m}) - \xi(0;\mathbf{m})$, we have

$$\tilde{\xi}(x;\mathbf{m}) = \begin{cases} c(m_k) + c'(m_k)(x-m_k) + c'(m_1)m_1 - c(m_1), & x \in [a_k,b_k] \\ c(x) + c'(m_1)m_1 - c(m_1), & \text{otherwise} \end{cases}$$

Distribution $G(x;\mathbf{m},v)$ is now defined by $\Phi_v(G(x;\mathbf{m},v)) = \min\{\tilde{\xi}(x;\mathbf{m}), v_1\}$ and satisfies the identities [**D.4**] and [**D.6**], as well as [**D.5**] for $k > 1$. For the derivative with respect to m_1, the identity takes the form

[D.7] $$\Phi_v'(G(x;\mathbf{m}^*,v))G_{m_1}(x;\mathbf{m}^*,v) = \tilde{\xi}_{m_1}(x;\mathbf{m}^*) = \begin{cases} c''(m_1^*)x, & x \in [0,b_1] \\ c''(m_1^*)m_1^*, & \text{otherwise} \end{cases}$$

With these modifications, the first term in the sum over k in [**D.2**] no longer integrates to zero. Instead, we have

$$\begin{aligned}
&-\int_0^{\bar{x}(\mathbf{m}^*,v)} \sum_{k=1}^{K} G_{m_k}(x;\mathbf{m}^*,v)\frac{\mathrm{d}m_k^*}{\mathrm{d}\delta}\,\mathrm{d}x \\
&= -c''(m_1^*)\frac{\mathrm{d}m_1^*}{\mathrm{d}\delta}\left[\int_0^{b_1}\frac{x}{c'(m_1^*)}\,\mathrm{d}G(x;\mathbf{m}^*,v)+\int_{b_1}^{\bar{x}(\mathbf{m}^*,v)}\frac{m_1^*}{c'(x)}\,\mathrm{d}G(x;\mathbf{m}^*,v)\right].
\end{aligned}$$

The sign of this term is determined by the sign of the derivative $\mathrm{d}m_1^*/\,\mathrm{d}\delta$, ie, by the direction of the equilibrium adjustment of the first mass point. The location of this mass point determines the equilibrium rent $\pi^* = c'(m_1^*)m_1^* - c(m_1^*)$. Recall that in Case 1 where $\xi_v^*(0) = 0$ and the agents earn zero rents these adjustments did not matter because all derivatives G_{m_k} integrate to zero.

The integral of $-G_\delta$ in [D.2] has the same structure as in Case 1 and is, therefore, positive. Thus, in order to show that the equilibrium expected effort increases, it is sufficient to show that $\mathrm{d}m_1^*/\,\mathrm{d}\delta \leqslant 0$. This is shown in Lemma 7.4 for *bottom-reducing* transfers in any equilibrium such that $H_{1m}(m_1^*, v) < 0$, where $H_k(m_k, v) = \int_{a_k}^{b_k}(x - m_k)\,\mathrm{d}G(x;\mathbf{m}, v)$. As we know, there is a possibility for multiple equilibria, ie, multiple values of m_1^*; however, these equilibria are ranked by the FOSD order in output in the same way as in the convex-concave case. Indeed, $\tilde{\xi}(x;\mathbf{m})$ increases in m_1. Sufficient conditions for the uniqueness of the equilibrium value of m_1^* are the same as well. We conclude that bottom-reducing transfers always raise the equilibrium effort in the equilibrium with the lowest value of m_1^*, which ultimately becomes a unique value. This implies the Main Theorem. □