I analyze a dynamic bargaining game in which the resources each group obtains today determine its relative agenda-setting power tomorrow. Although every member of the society possesses voting rights collectively to prevent further concentration of proposal power, this bargaining process of endogenous proposal power ends up with complete domination of a single group that possesses all of the resources and proposal power. Further, regardless of the number of groups and the distribution of resources and proposal power in the initial state, this extreme form of domination emerges at most in three rounds of bargaining. Once a group is deprived of proposal power, it approves every proposal since no other group will assign any positive portion of the dollar and proposal power in the future. Thus once there are enough number of groups that are willing to vote for everything, any recognized proposer immediately takes the whole resources. Even when the proposer is not able to directly propose to assign all of the resources to itself, the proposer’s interest lies in eliminating as many groups as possible, especially the groups that have a greater portion of the resources and better prospects in the competition toward the perpetual domination.

**keywords:** Proposal power, dynamic bargaining, Markov perfect equilibrium, concentration of power, inequality
1 Introduction

The influence of money in politics has long been at the center of public and academic discourse around the world. Resource-rich elites at times employ blatantly corrupt practices such as vote buying or turnout buying (Bratton, 2008; Brusco, Nazareno and Stokes, 2004; Cox and Kousser, 1981; Nichter, 2008; Stokes, 2005; Vicente, 2014). In places where vote buying or turnout buying practices prevail, the individuals or groups possessing more money have a greater capability to buy votes and gain more seats in elected political bodies. These bodies can in turn shape the rules governing economic transactions and distribution of the means of production and economic surpluses. Economic elites in other places seek more subtle means to influence political decision-making processes such as campaign contributions or lobbying. In the US, the 2010 Supreme Court decision in *Citizens United* allowed independent groups unlimited spending in federal elections on the basis of free speech. Following this decision, campaign contributions by the top 0.01 percent of the voting age population accounted for over 40 percent of the total contributions even when excluding the contributions by social welfare nonprofit organizations in the 2012 elections (Bonica et al., 2013).\(^1\) The normative and positive implications of the Supreme Court ruling are highly controversial, and its political consequences are to be seen in later elections. Considering the costs and impacts of television and radio ads in recent elections, however, it is hard to deny that the resource-rich have advantages in making their voices heard in electoral arena and shaping agenda in elected political bodies. Some raise concerns about the increasing influence of the economic elites in the agenda-setting process for it undermines the fairness and legitimacy of the political system as a whole. Some people use campaign donations and lobbying to buy access to the legislative process and get their concerns on the agenda, which probably results in policies in favor of the big spenders. Others argue that the eventual policies may wind up being reasonably representative of the views of ordinary citizens. The democratic system is after all set up so that each group has to persuade a majority of people to support their proposals. Ordinary citizens, despite the differential access to the agenda-setting process, may be able to counterbalance such biases occurring at the agenda-setting stage because they have the power to reject unfavorable policies at the voting stage.

In this paper, I investigate the long-run distribution of resources and proposal power in the presence of resource-power interdependency by presenting a formal model in which members of a society collectively make a decision on how to distribute a limited amount of resources through a bargaining process, and this resource distribution determines each member’s agenda-setting power in the next round of the decision-making process. More specifically, I study an infinite horizon divide-the-dollar game among three or more players in which the status quo division is determined endogenously and future recognition probabilities evolve over time depending on the current bargaining outcome. Equality in voting rights and endogenously arising inequality in proposal rights of the players is the primary novelty of the model. Although players’ proposal power changes during the process of the bargaining game, all players maintain equal voting rights regardless of their access to resources and proposal power. In principle the players collectively possess an instrument—votes—to prevent the implementation of undesirable policy proposals even if they do not have proposal power. “One person, one vote” is perhaps one of the most important principles of modern democracy. At the same time most citizens do not have agenda-setting power. The legitimacy of representative democracy is then guaranteed by the belief that ordinary citizens can influence government policies by means of votes. This model thus provides an opportunity to reevaluate one of the key principles of representative democracy by examining the value of voting rights in the absence of proposal power.

To preview the results, granting equal voting rights does not enhance one’s political standing if it is not accompanied by proposal power. In any equilibrium, proposers in each period attempt to extract as much resources as possible by securing the minimal support necessary and eliminate other players from the subse-

\(^1\)These social welfare nonprofits, not required to disclose the identity of donors, spent over 250 million dollars in the 2012 elections according to the Center for Responsive Politics (https://www.opensecrets.org).
quent bargaining process. More surprisingly, it takes at most three rounds until a single group monopolizes all of the wealth and agenda-setting power regardless of the number of players and the initial distribution of resources and proposal power. Once such a monopoly emerges and possesses all of the resources and proposal power, all players unanimously accept continuing domination of that group since they cannot expect to recover proposal power or receive a positive share of the dollar in the future. Introducing more players with voting rights but no proposal power may further facilitate transition to the extreme domination.

The model of endogenous proposal power lies in the theoretical literature on sequential bargaining originated from Rubinstein (1982) and Baron and Ferejohn (1989). Most models in this line of research agree on the advantage of the agenda-setter. Within a Baron-Ferejohn framework, Kalandrakis (2006) shows that any level of expected payoffs, which he perceives as political power, can be supported in equilibrium by properly distributing proposal power irrespective of voting power. Eraslan (2002) establishes the uniqueness of stationary equilibrium payoffs in the Baron-Ferejohn model and shows that a player with greater proposal power attains (weakly) better equilibrium payoffs if all players are equally patient.

More specifically this study contributes to the growing literature on dynamic bargaining with endogenous status quo (Anesi and Seidmann, 2014; Baron, 1996; Baron and Bowen, 2013; Bernheim, Rangel and Rayo, 2006; Bowen and Zahran, 2012; Bowen, Chen and Eraslan, 2014; Diermeier and Fong, 2011; Duggan and Kalandrakis, 2012; Kalandrakis, 2004, 2010; Nunnari, 2011; Penn, 2009; Richter, 2014; Zapal, 2014). Duggan and Kalandrakis (2012) establish a general existence result of Markov equilibria by assuming small perturbations on preferences and status quo, which is necessary to invoke fixed point argument. In the model of endogenous proposal power, I prove the existence of Markov perfect equilibria without assuming stochastic shocks. The presence and the extent of the agenda-setter’s power is also a central theme in the aforementioned studies. Kalandrakis (2004, 2010) finds that the bargaining process with endogenous status quo eventually invites a per-period dictator—the agenda setter. The advantage of the agenda-setter is not always unconstrained in these models. In a model of one-dimensional policy space, Baron (1996) demonstrates that a proposer may compromise her policy preference to ensure stability of a policy and the policy converges to the median in the long run. Diermeier and Fong (2011), in a setting with a persistent agenda setter and a discrete policy space, show that the agenda-setter’s discretion is limited since other players anticipate further exploitation by the agenda setter and, hence, protect each other so that other players do not cooperate with the persistent proposer’s future plans. Bowen and Zahran (2012) finds that almost equal allocations are possible and proposers in each period may compromise in equilibrium. Risk-averse players have incentives to advocate continuation of larger coalitions, which can be supported if excessively unequal allocations trigger a punishment scheme that cannot return to the compromise scheme. The long-run outcome of the game depends on the level of patience players have and the initial state. Allowing players to waste a portion of the pie in the course of play, Richter (2014) constructs a Markov equilibrium in which equal division can be supported as the only long-run outcome regardless of the initial state. Players distinguish whom to punish and reward depending on their status quo shares in each state. In each period the proposer punishes those who have more than a fair share of the dollar and, more importantly, restrains oneself from extracting more pie anticipating the same punishment from future proposers, which completely neutralizes agenda-setter’s advantages.

The model presented in this paper advances this literature by incorporating the evolution of agenda-setting power and its dependency on the resources under disposal of each player. The players’ proposal power evolves over time as a result of the current agreement on the resource allocations. In the existing bargaining models with endogenous status quo, the rules selecting a proposer have been treated as exogenous and assumed to be fixed throughout the whole bargaining process. Contrast to the equilibrium compromise of proposers found in the aforementioned models, the model of endogenous proposal power predicts the

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2 Working paper by Jeon (2014) considers a similar model in which both proposal and voting power of the players endogenously change over time assuming that all players are perfectly patient and shows that a permanent tyranny of a single player or a permanent oligarchy of two players emerge in the long run.
emergence of a dictator, regardless of discount factors and initial states, in equilibrium as in Kalandrakis (2004, 2010) but a permanent dictator instead of a rotating one.

As to the endogenous evolution of proposal power, there are several studies noteworthy. McKelvey and Riezman (1992) deal with the connection between the seniority system in the U.S. legislature and the incumbent advantage. Eguia and Shepsle (2012) and Muthoo and Shepsle (2010) extend McKelvey and Riezman’s model and provide explanations on the endogenous emergence of the seniority system that gives greater proposal power to seniors. Informed about the seniority system voters always reelect the incumbent, and it again gives incentive to legislators to maintain the seniority system. Yildirim (2007, 2010) adds a pre-bargaining stage in which players compete to obtain greater proposal power prior to the bargaining stages of the game. He finds that a persistent recognition rule results in a more unequal outcome than the transitory rule. In Yildirim’s model the bargaining game ends once there is an agreement, whereas the bargaining process resumes in every period with an endogenously evolving status quo and proposal rule in my model. Cotton (2012) examines two endogenous proposal power models: one in which a successful agenda setter automatically maintains power in the next period; and another in which a proposer needs majority support to remain in power. He finds that the model featuring voting on the identity of the agenda setter yields the most equitable outcome and the outcome is more equitable if the players are farsighted. His model differs from mine in that the status quo policy is exogenous and fixed.

The rest of the paper is organized as follows: Section 2 introduces the bargaining environment, necessary notation and equilibrium notion; in Section 3, I analyze the equilibrium dynamics assuming that equilibrium exists. Equilibrium existence is established in Section 4; and then I summarize the results and conclude. I provide sketches of the proofs in the main text and the formal proofs are relegated to Supporting Information.

2 Model

There is a finite set \( I = \{1, \ldots, n\} \) of players who collectively divide a dollar each period in an infinite time horizon, \( t = 1, \ldots, \infty \). The players in the model can be seen as individuals or groups in a society. Let \( X \) be the set of all feasible divisions of the dollar: \( X = \{ x \in \mathbb{R}^n | \sum I x_i = 1, x_i \geq 0, \forall i \in I \} \). In each period \( t \), the bargaining environment is summarized by two parameters \((x^{t-1}, p^t)\), where \( x^{t-1} \) is the status quo division of the dollar and \( p^t \) is a vector representing recognition probabilities of the players.

The timing of the game is as follows: At the beginning of each period \( t \), a player is chosen as a proposer according to the recognition probability \( p^t \) and makes a proposal \( y \in X \). Observing this proposal, all players simultaneously vote to accept or reject the proposal by simple majority rule. Every player has one vote in each period. If the proposal \( y \) obtains more than a majority of supporting votes, it is implemented as the policy of that period: \( y = x^t \). Otherwise, the status quo division from the previous period is implemented instead: \( x^{t-1} = x^t \). At the end of each period \( t \), the players receive a stage payoff \( u_i(x^t) = x^t_i \). The bargaining in period \( t + 1 \) begins in a new environment \((x^t, p^{t+1})\), and this process continues infinitely.

Regarding the evolution of proposal power, I assume that \( p^{t+1} = x^t \). The policy chosen in period \( t \) determines the proposal power of the players in the next period \( t + 1 \). Thus this model captures the situations in which the players who obtain greater resources today enjoy greater proposal power tomorrow. The players discount future flows of income with a common factor \( \delta \in [0, 1) \).

In this game, the players’ decisions involve two kinds of strategies. Let \( \sigma_i \) be a pair of player \( i \)'s proposal strategy \( \mu_i \) and voting strategy represented by an acceptance set \( A_i \): \( \sigma_i = (\mu_i, A_i) \). For each history up to period \( t - 1 \), including the identity of proposers, their proposals, and all the players’ voting records, \( \mu_i \) prescribes player \( i \)'s proposals he would make if recognized in period \( t \). Similarly, for each history up to period \( t - 1 \), the identity of the period \( t \) proposer, and a proposal already made in period \( t \), \( A_i \) dictates whether player \( i \) would vote for or against a proposal. In the following analysis I focus on stationary Markov strategies. In stationary Markov strategies, players choose the same action whenever the state variables are
identical irrespective of how the game has evolved up to the state and the time index $t$. In the bargaining environment introduced above, all the players care only about the distribution of wealth in the status quo division $x^{t-1}$ and the recognition probability $p_t$ in each period. Since the status quo division $x^{t-1}$ completely determines $p_t$, define period $t$ state by $s^t = x^{t-1}$. The set of all states $S = X$. Denote the set of probability distributions over $X$ by $\Delta X$. Player $i$’s Markov strategy is a pair $\sigma_i = (\mu_i, A_i)$, where $\mu_i : S \rightarrow \Delta X$ specifies player $i$’s probability of proposing any divisions of the dollar in each state $s$ and $A_i : S \Rightarrow X$ dictates the set of divisions of the dollar for which player $i$ will vote for in each state. Regarding proposal strategies, let $\mu_i(y|s)$ denote the probability player $i$ assigns to $y \in X$ in state $s$. I assume that the support of $\mu_i$ is finite for all $i$ and $s$. In order to ease notational burden, I use $\mu_i(s) = y$ instead of $\mu_i(y|s) = 1$ if player $i$ proposes $y$ with probability 1.

Since the players are not ex ante different from each other, I further focus on symmetric strategies that no player differentiates other players merely due to their names and no two players behave differently merely due to their names. In other words, if two player $i$ and $j$ have the same status quo shares and proposal power in a state, their proposals and acceptance sets are identical except for their names, and all the other players treat $i$ and $j$ identically. Formally, for a one-to-one and onto function $\phi : I \rightarrow I$ and for any $x \in X$, let $\hat{\phi} : X \rightarrow X$ be a permutation under $\phi$ such that $\hat{\phi}(x_i) = (x_{\phi(1)}, x_{\phi(2)}, \ldots, x_{\phi(n)})$. A Markov strategy profile $\sigma$ is symmetric if for any $x$ and its $\hat{\phi}$-permutation $\hat{\phi}(x)$, $\sigma_i(\hat{\phi}(x)) = \sigma_{\hat{\phi}(i)}(x)$ for every $i$. Let $\sigma = \prod_{i=1}^n \sigma_i$ be any symmetric stationary Markov strategy profile and $\Sigma$ be the set of all such strategy profiles.

Since a Markov strategy profile $\sigma$ dictates how the bargaining game will unfold beginning from any state, we can define player $i$’s continuation value for any state $s$ in $\sigma$.\footnote{For notational simplicity, I use $\mu_i$ and $A_i$ for $i$’s proposal strategy and voting strategy in $\sigma$. For a strategy profile $\hat{\sigma}$, I denote the proposal strategy as $\hat{\mu}_i$ and voting strategy as $\hat{A}_i$.} Player $i$’s continuation value $v_i^\sigma(s)$ is his ex ante expected payoff from state $s$ before the identity of the proposer is known. In order to formally define $v_i^\sigma(s)$, I introduce additional notation. A set $L$ of players is a decisive coalition if and only if the members of $L$ collectively constitute a majority. If there are odd number of players, any $L$ such that $|L| \geq \frac{n+1}{2}$ is a decisive coalition, and if $n$ is even, any $L$ such that $|L| \geq \frac{n}{2} + 1$ is a decisive coalition. By collectively voting for or against a proposal, the members of a decisive coalition can pass or fail a proposal. Denote the set of all decisive coalitions by $\mathcal{L}$. The social acceptance set, $A(s)$, is the set of all proposals that can be passed in $s$: $A(s) = \bigcup_{L \in \mathcal{L}} \bigcap_{i \in L} A_i(s)$. For each $y \in X$, let $\mu(y|s)$ be the probability that $y$ is proposed in $s$: $\mu(y|s) = \sum_{y_i \in \mu_i(y|s) > 0} \mu_i(y|s)(y_i + \delta v_i^\sigma(y_i))I_{A_i(s)}(y_i) + (s_i + \delta v_i^\sigma(s_i))I_{X \setminus A_i(s)}(y_i)$.

Define player $i$’s continuation value from $s$ in $\sigma$ by

$$v_i^\sigma(s) = \sum_{y \in \mu(y|s) > 0} \mu(y|s) \left[ (y_i + \delta v_i^\sigma(y_i))I_{A_i(s)}(y_i) + (s_i + \delta v_i^\sigma(s_i))I_{X \setminus A_i(s)}(y_i) \right]$$

where $I_A(y_i) = 1$ if $y_i \in A$ and 0 otherwise. Consider any $y \in X$ that will be proposed with probability $\mu(y|s)$ in $s$ in $\sigma$. If it is in the social acceptance set, it gives player $i$ a short-term utility $y_i$ for that period, and the bargaining process resumes in the next period in the new state $y$ from which player $i$ anticipates to receive $v_i^\sigma(y_i)$. If $y$ is not in the social acceptance set, player $i$ receives the status quo share $s_i$, and the players resume the next period bargaining process in the same state $s$ which gives player $i$ continuation value $v_i^\sigma(s)$. Using the continuation value $v_i^\sigma(s)$, define player $i$’s expected utility from any division $x \in X$ in $\sigma$ by

$$U_i^\sigma(x) = x_i + \delta v_i^\sigma(x).$$

Facing any proposal $x$, player $i$ considers her immediate (short-term) utility from $x$ and the continuation value from the new state $x$. Note that the sum of all players’ expected utilities is $\frac{1}{1+\delta}$ in any strategy profile $\sigma$. Let $\mathcal{U}$ denote the set of all possible expected utility combinations in $S$ and $\Sigma$. That is, $\mathcal{U} = \{U^\sigma(s) : \exists s \in S \text{ and } \exists \sigma \in \Sigma \}$, where $U^\sigma(s) = \sum_{i=1}^n U_i = \frac{1}{1+\delta}$ and $U_i \geq 0$, $\forall i \in I$. 


I seek subgame perfect equilibria in symmetric stationary Markov strategies with two additional properties. First, I require that all players use stage-undominated voting strategies. Due to the pivotality issue, most majority voting games have uninteresting equilibria. As an example, suppose that $n \geq 3$ and all players except a single player strictly prefer proposal $y$ to the status quo $s$. Given a strategy profile in which all players vote against $y$ in $s$, no player can change the voting outcome alone. Hence, no player has incentives to deviate from the given strategy, and it constitutes an equilibrium. To exclude these implausible equilibria, I require that all players make voting decisions as if they are pivotal and use stage-undominated voting strategies (Baron and Kalai, 1993). In every state $s$, player $i$’s acceptance set should not include the divisions that give $i$ strictly worse expected utility than the status quo and should include all the divisions that is weekly better than $s$. Second, I require that no player makes a proposal that will be rejected. It is not restrictive since making a socially unacceptable proposal is practically the same as proposing the status quo division. This assumption simplifies the analysis but has no effect on equilibrium existence and characterization.

**Definition 1.** A symmetric stationary Markov perfect Nash equilibrium in stage-undominated voting strategies is a Markov strategy profile $\sigma^* = ((\mu^*_i, A^*_i))_{i=1}^n$, such that $\forall i \in I$, $\forall s \in S$ and $\forall$ permutations $\hat{\phi} : X \to X,$

$$\sigma^*_i(\hat{\phi}(s)) = \sigma^{\phi(i)}_i(s) \tag{1}$$

$$A^*_i(s) = \{x \in X | U^*_i(x) \geq U^*_i(s)\} \tag{2}$$

$$\mu^*_i(y|s) > 0 \Rightarrow y \in \arg\max_x \{U^*_i(x) | x \in A^*_i(s)\} \tag{3}$$

Condition (1) is on symmetry of strategies, and condition (2) is on the stage-undominated voting strategies. Condition (3) is an optimality condition for the players’ proposals and also says that all proposals are in the social acceptance set.

### 3 Equilibrium Dynamics

In this section I describe the players’ equilibrium proposals, voting behaviors and the long-run equilibrium outcome of the dynamic bargaining game with endogenous proposal power. The most striking result is the emergence of permanent tyranny, extreme domination of a single player, within at most three rounds. Once in place, the tyrant obtains unanimous approval in all subsequent periods and never shares her wealth and proposal power with other players.

In equilibrium every recognized proposer in every state forms a minimum-cost winning coalition. Every proposal is approved by a set of coalition partners whose collective demand is the lowest. Since each player votes for a proposal insofar as the new proposal guarantees the same expected utility with the status quo allocation, every proposer then gives one’s minimum-cost winning coalition partners exactly their status quo expected utilities but no more. Proposition 1 formally describes the set of equilibria. For each player $i$, let $\mathcal{L}_i$ be the set of decisive coalitions that include $i$ and $\hat{\mathcal{L}}_i$ be the collection of sets of player $i$’s decisive coalition partners. That is, $\mathcal{L}_i = \{L \in 2^I \cup \{i\} | i \in \mathcal{L}_{+i} \text{ and } i \notin L\}$. 

Proposition 1. The set of equilibria can be described as follows. For all \( s \in S \) and \( i \in I \),

\[
i \text{proposes } y^{i|\eta} \in X \text{ with probability } \mu_i(y^{i|\eta}|s),
\]

and \( y \in A_j(s) \iff U_j(y) \geq U_j(s) \text{ for all } j \in I, \)

where

\[
U_i(y^{i|\eta}) = \frac{1}{1-\delta} - \sum_{j \neq i} U_j(y^{j|\eta})
\]

\[U_j(y^{j|\eta}) = U_j(s) \text{ for all } j \in L_{\phi} \]

\[U_k(y^{j|\eta}) = 0 \text{ for all } k \notin L_{\phi} \cup \{i\},\]

\[
U_i(s) = s_i + \delta \left( s_i \sum_{y^{i|\eta} \mu_i(s|s)>0} \mu_i(y^{i|\eta}|s)U_i(y^{i|\eta}) + \sum_{j \neq i} s_j \sum_{y^{j|\eta} \mu_j(s|s)>0} \mu_j(y^{j|\eta}|s)U_j(y^{j|\eta}) \right)
\]

and

\[
\|\eta\| > 0 \Rightarrow L_{\phi} \in \arg \min_{L \in \bar{L}} \left[ \sum_{j \in L} U_j(s) \right].
\]

In the strategy profile described in Proposition 1, all players accept another player’s proposal if and only if the proposal is weakly better than the status quo. Given these voting strategies, any proposer finds the sets of decisive coalition partners whose collective sums of status quo expected utilities are the smallest, and she makes a proposal that guarantees the status quo expected utilities to its coalition partners and nothing to the players unnecessary to form a decisive coalition. Since a proposer takes the remainder of the expected utility after paying her coalition partners’ demands, she does not propose any allocation that gives a higher expected utility than necessary to any of her coalition partners or a positive expected utility to other players not included in the chosen set of decisive coalition partners. When there are multiple sets of coalition partners whose sums of status quo expected utilities are the lowest, the proposer is indifferent between selecting any such sets of coalition partners. However, the mixing probability among these sets of coalition partners must generate the status quo expected utility in (9).

There are two noteworthy points in Proposition 1. First, although every proposer forms minimum-cost winning coalitions when recognized to propose, these minimum-cost winning coalitions may not be of minimum-winning in terms of their sizes. Second, the proposals associated with each minimum-cost winning coalition remains the same. As seen in (6)-(8), \( U(y^{i|\eta}) = U(y^{i|\eta'}) \) for any distinct allocations \( y^{i|\eta} \) and \( y^{i|\eta'} \) that are associated with the same set of minimum-cost winning coalition partners \( L_{\phi} \). Thus any relative probability distribution within these set of allocations results in the same status quo expected utility vector \( U(s) \).

Lemma 1 states equilibrium voting behaviors of players having no wealth and proposal power in any given state, and it plays a key role in the subsequent analysis.

Lemma 1. For all \( \delta \in [0, 1) \) and \( s \in S \), \( U_i(s) = 0 \) if and only if \( s_i = 0 \); and \( A_i(s) = X \) if and only if \( s_i = 0 \).

In words, the players who currently have no access to resources and no power to propose an alternative allocation remain deprived in all subsequent periods despite their equal voting rights. In each state, a recognized proposer distributes positive shares of the dollar and positive expected utilities to a subset of other players as a compensation for these players’ possible gains from rejecting a proposal and staying in the current state. The players having no resources in the current state, who I refer to as deprived players, do not expect to receive any positive share of the dollar immediately and recover its proposal power in the future by rejecting a proposal. Being aware of the situation of deprived players, active players, who currently have a share of the wealth and proposal power, have no incentive to share the wealth and future proposal power.
with deprived players in their proposals. Thus the expected utility of deprived players is zero and these players vote in favor of every proposal regardless of the content out of indifference in equilibrium.

For any natural number \( k \leq n \), let \( \Delta_k \) be the set of states with \( k \) active players. For instance, \( \Delta_1 \) is the set of tyrannical states and \( \Delta_n \) is the set of states in which all players have a positive chance to be a proposer. Obviously, \( S = \bigcup_{k=1}^n \Delta_k \), and \( \Delta_k \cap \Delta_{k'} = \emptyset \), for any \( k \neq k' \). For notational convenience I reserve \( l \) and \( m \) to refer to natural numbers in particular ranges: \( 1 \leq l \leq \frac{n+1}{2} \) and \( \frac{n+1}{2} < m \leq n \).

From Lemma 1 we can straightforwardly infer the equilibrium proposals by active players in a subset of states. In any state, active players obtain all deprived players’ votes without any cost. Accordingly, if there are a critical number of deprived players to form a decisive coalition, any recognized active player can form a decisive coalition only with deprived players without paying any cost, and pass a tyrannical proposal immediately.

**Lemma 2.** For all \( \delta \in [0, 1) \) and \( s \in \Delta_l \), every active player proposes to take the entire dollar and \( U_l(s) = \frac{1}{1-\delta} \) for all \( i \in I \) in equilibrium.

In any state \( s \in \Delta_l \), a proposer and the deprived players collectively make a decisive coalition. Since the deprived players will vote for any proposal, the unique optimal proposal for any proposer is to take the entire dollar by oneself. As an example, consider a state with four active players in a seven-player game. In such a state one of the four active players is recognized as a proposer and proposes to take the entire dollar by oneself, which is approved by the proposer and the three deprived players. If the initial state is tyranny, the tyrant proposes the same tyrannical allocation in every period and all other players, who are deprived, vote in favor of it. Note that all active players’ proposals are unique, and the equilibrium expected utility vector \( U(s) = (U_1(s), \ldots, U_\mu(s)) \) is also unique for every \( s \in \Delta_l \) in equilibrium.

Now we turn to the states in \( \Delta_m \). In such states, active players cannot form a decisive coalition even if they mobilize all deprived players. Thus they need to satisfy some other active players, called costly coalition partners, in order to obtain approval of a decisive coalition. Suppose that \( n \) is even and the current state \( s' \) is in \( \Delta_m \). Then there are \( n-m \) deprived players, and any proposer in \( s' \) needs at least \( m - \frac{n}{2} \) other active players’ votes to pass a proposal. Every proposer whose aim is to form a minimum-cost winning coalition selects exactly \( m - \frac{n}{2} \) other active players whose status quo expected utility is the lowest and gives them exactly their status quo expected utility in equilibrium. Including additional active players to the coalition or giving more than the status quo expected utility to any of the selected coalition partners reduces the proposer’s payoff from her proposal. Hence any allocation proposed with positive probability in \( s' \) gives positive expected utilities to \( m - \frac{n}{2} + 1 \) players in equilibrium, and the state in the next period is in \( \Delta_m - \frac{n}{2} + 1 \). Since \( \frac{n}{2} + 1 \leq m \leq n \), the number of active players in \( s' + 1 \) is less than \( \frac{n}{2} \) if \( n < m \) and in \( \frac{n}{2} + 1 \) if \( m = n \). If \( n \) is odd, the number of necessary costly coalition partners is \( m - \frac{n+1}{2} \), and the number of active players in the next period is \( m - \frac{n+1}{2} \). Lemma 3 summarizes the above discussion and the proof is omitted. Recall that \( 1 \leq l \leq \frac{n+1}{2} \) and \( \frac{n+1}{2} < m \leq n \).

**Lemma 3.** Suppose that the current state is \( s' \in \Delta_m \). In equilibrium, the state in the next period is \( s' + 1 \in \Delta_{m-1} \) if \( n \) is even and \( m = n \); and \( s' + 1 \in \Delta_l \) otherwise.

Lemma 2 and 3 together imply that in equilibrium a permanent tyrant, who monopolizes the entire wealth and proposal power, emerges within at most three rounds regardless of the initial state. If the initial state is in \( \Delta_l \), the transition to tyranny is immediate. If the initial state is in \( \Delta_m \), except for the case in which \( n \) is even and \( m = n \), the second period state is in \( \Delta_l \), which turns into tyranny in the third period regardless of the identity of the recognized proposer. Finally, if \( n \) is even and all players have positive proposal power

\[ \text{If } s \in \Delta_m \text{, any active player can make a decisive coalition together with all deprived players. Roughly speaking, there are less active players than deprived players except when the number of active players is exactly } \frac{n+1}{2} \text{ if } n \text{ is odd and } \frac{n}{2} \text{ if } n \text{ is even. If } s \in \Delta_m, \text{ there are more active players than deprived players.} \]
in the initial state, there will be exactly \( \frac{n}{2} + 1 \) active players in the second period, which leads to a state with two active players in the third period, and then we have a tyrant in the fourth period.

**Corollary 1.** In equilibrium, a permanent tyrant emerges within at most three rounds regardless of the number of players, discount factors and initial states.

If the players use current resources to enhance their future proposal power, permanent tyranny is inevitable. Until a permanent tyrant is in place, the proposer in each period selects the cheapest or the weakest players as coalition partners and exploits more powerful players. On the one hand, possessing a greater share of the wealth is an advantage since that player is more likely to be a proposer. On the other hand, having more status quo share may be a disadvantage because it increases the probability to be eliminated by other players in case the recognized proposer is not oneself. Accordingly, it is possible that a more resource rich and more powerful player may be less likely to survive in the end. The next lemma answers the question pertaining to the advantage and disadvantage of having a greater share of the dollar in any state.

**Lemma 4.** In equilibrium, \( s_i \geq s_j \) implies \( U_i(s) \geq U_j(s) \) in all states and for all discount factors.

Recall that a player’s expected utility is one’s long-term payoff in expectation. Since the players anticipate a forthcoming tyranny in equilibrium, each player’s expected utility roughly reflects how likely one is to survive and become a tyrant in the end. Lemma 4 says that having a greater status quo share \textit{ex ante} never hurts a player. Also no proposer intentionally takes a smaller share of the dollar just to increase the likelihood to be included in other players’ coalitions in the next period.

To give an intuition behind the result, suppose on the contrary that player 1’s status quo share is greater than player 2’s share but player 2’s expected utility is higher than player 1’s. As seen in the definition of expected utility, each player’s status quo expected utility is determined by one’s share of the dollar, the recognition probability, the expected utility from one’s own proposal, the probability to be chosen as a coalition partner by other players and the expected utility from other players’ proposals. Since player 1’s current share is greater and is more likely to be a proposer than player 2, their expected utility difference must come from other players’ choices of coalition partners. In other words, the probability that player 2 is chosen as a coalition partner of other players needs to be higher than player 1’s if player 2’s expected utility were higher. However, all other players that choose player 2 as a coalition partner must also include player 1 in the coalition in equilibrium since player 1 is a less costly coalition partner than player 1 by supposition. Now player 1 is not only more likely to be a proposer but also more likely to be included in other players’ coalition than player 2. Therefore we reach a contradiction that player 1’s expected utility is higher than player 2’s.

A higher-share player’s \textit{ex ante} advantage disappears once another player is selected as a proposer simply because this higher-share player is a more expensive potential coalition partner to other players. Lemma 4, however, does not imply that the players having the lowest status quo shares are always included in other players’ minimum-cost winning coalitions. For instance, if the difference of two players’ status quo shares is very small and other players all choose the smaller-share player and not the higher-share player as their coalition partner, the smaller-share player’s expected utility may become higher than the higher-share player’s. Then other players’ proposals are not optimal since they are choosing a more expensive coalition partner instead of a less expensive one. The situation in which all the other players then choose the higher-share player as a coalition partner instead of the lower-share player obviously cannot be an equilibrium behavior since then the higher-share players’ expected utility in this case will be even higher than the smaller-share player’s. Equilibrium proposals need to involve choosing different coalition partners probabilistically depending on the status quo allocation and discount factor. The following example demonstrates the needs for mixed proposals for different discount factors.

**Example 1.** Suppose that \( n = 3 \) and \( s = \left( \frac{5}{10}, \frac{3}{10}, \frac{2}{10} \right) \).
(i) Let \( \delta = 0 \). In equilibrium, player 1 and 2 choose player 3 as a coalition partner and propose \((\frac{8}{10}, 0, \frac{2}{10})\) and \((0, \frac{8}{10}, \frac{2}{10})\), respectively. Player 3 chooses player 2 as a coalition partner and proposes \((0, \frac{3}{10}, \frac{7}{10})\). By \( \delta = 0, U(s) = s \).

(ii) Let \( \delta = \frac{1}{2} \). Suppose that players 1 and 2 choose player 3 and player 3 chooses player 2 as a coalition partner. Then

\[
U_1(s) = \frac{5}{10} + \frac{5\delta}{10} \left[ \frac{1}{1 - \delta} - U_3(s) \right] \\
U_2(s) = \frac{3}{10} + \frac{3\delta}{10} \left[ \frac{1}{1 - \delta} - U_3(s) \right] + \frac{2\delta}{10} U_2(s) \\
U_3(s) = \frac{2}{10} + \frac{2\delta}{10} \left[ \frac{1}{1 - \delta} - U_2(s) \right] + \frac{8\delta}{10} U_3(s).
\]

Solving the equations yields \( U(s) = (\frac{6}{7}, \frac{4}{7}, \frac{4}{7}) \). Thus player 1 and 2 choose player 3 as a coalition partner and propose \((\frac{6}{7}, 0, \frac{4}{7})\) and \((0, \frac{6}{7}, \frac{4}{7})\), respectively. Player 3 chooses player 2 as a coalition partner and proposes \((0, \frac{4}{7}, \frac{4}{7})\) in equilibrium. These are the unique equilibrium proposals for the players in \( s \). If player 1 chooses player 2 as a coalition partner with any positive probability, player 2’s expected utility increases and player 3’s expected utility decreases than the given \( U(s) \), which makes player 1’s choice of player 2 as a coalition partner not optimal. The equilibrium proposals come from Lemma 2 that \( U(y) = \frac{y}{1-\delta} = 2y \) for \( y \in \Delta \) and from the fact that every proposer gives one’s coalition partner exactly the partner’s current expected utility.

(iii) Let \( \delta = \frac{2}{3} \). Suppose that player 1 and 2 choose player 3 and player 3 chooses player 2 as a coalition partner, and denote the proposals involving such coalition plans by \( \tilde{\mu}(s) \). Using above equations, we obtain \( U^\sigma(s) = (\frac{39}{34}, \frac{27}{34}, \frac{36}{34}) \). Then player 1’s proposal is not optimal since his coalition partner, player 2, is more expensive than player 2, and \( \tilde{\mu}(s) \) cannot be equilibrium proposals in \( s \).

Suppose that player 1 chooses player 2 with probability \( q \) and player 3 with probability \( 1-q \) and Player 2 and 3 choose each other as a coalition partner with probability \( 1 \), and denote the proposal profile in \( s \) involving such coalition plans by \( \mu(s) \). Then,

\[
U_1(s) = \frac{5}{10} + \frac{5\delta}{10} \left[ \frac{1}{1 - \delta} - U_3(s) \right] \\
U_2(s) = \frac{3}{10} + \frac{3\delta}{10} \left[ \frac{1}{1 - \delta} - U_3(s) \right] + \frac{5q\delta}{10} U_2(s) + \frac{2\delta}{10} U_2(s) \\
U_3(s) = \frac{2}{10} + \frac{2\delta}{10} \left[ \frac{1}{1 - \delta} - U_2(s) \right] + \frac{5(1-q)\delta}{10} U_3(s) + \frac{3\delta}{10} U_3(s).
\]

Solving the equations using \( U_2(s) = U_3(s) \) yields \( U(s) = (\frac{12}{10}, \frac{9}{10}, \frac{9}{10}) \) and \( q = \frac{1}{5} \). Since the state in the next period is in \( \Delta \), each player’s expected utility in the next period is \( U(y) = \frac{y}{1-\delta} = 3y \) for any proposal \( y \). Thus player 1 proposes \((\frac{7}{10}, \frac{3}{10}, 0)\) with probability \( \frac{1}{5} \) and \((\frac{2}{10}, 0, \frac{3}{10})\) with probability \( \frac{4}{5} \). Player 2 and 3’s proposals are \((0, \frac{7}{10}, \frac{3}{10})\) and \((0, \frac{3}{10}, \frac{7}{10})\), respectively. Again, these are unique equilibrium proposals in \( s \) since otherwise each player chooses a more expensive player as a coalition partner.

### 4 Existence of Equilibria

In the previous section, I have shown that permanent tyranny endogenously emerges within at most three rounds in equilibrium while taking existence of equilibria for granted. I establish existence of equilibria in this section. There are multiple equilibria, but all equilibria are payoff equivalent in the sense that \( U^\sigma(s) = U^\sigma(s) \) for every state \( s \) in any two distinct equilibrium strategy profiles \( \sigma \) and \( \tilde{\sigma} \).
Since equilibrium proposals are obvious in all states in $\Delta_i$ by Lemma 2, the analysis below focuses on the states in $\Delta_m$. I first formulate a state game $\Gamma(s)$, which is a Baron-Ferejohn sequential bargaining game, for each $s \in \Delta_m$. In the state game $\Gamma(s)$, the players divide a good of total value $\frac{1}{1-\delta}$. Lemma 5 establishes the existence of symmetric stationary subgame perfect equilibria in undominated-voting strategies (SSPE) of $\Gamma(s)$. Lemma 6 shows that for any proposal $U$ in the support of SSPE proposal strategies in $\Gamma(s)$, there is an allocation $y \in X$ from which all players’ equilibrium expected utilities equal to the proposal $U$. Replacing the proposals $U$ with corresponding $y \in X$ in each $\Gamma(s)$ and taking the collection of SSPE proposal strategies of $\Gamma(s)$ for all $s \in \Delta_m$. Proposition 2 establishes the existence of symmetric stationary Markov perfect equilibria in stage-undominated voting strategies $\sigma^*$. Given that equilibrium $\sigma(s)$ corresponds to SSPE of the state game $\Gamma(s)$ for every state, the uniqueness of equilibrium expected utility $U = \{U(s)\}_{s \in S}$ follows from Eraslan (2002) and Eraslan and McLennan (2013).

In the description of the game $\Gamma(s)$, I assume that $n$ is even to avoid unnecessary complication and relegate the odd $n$ case to footnotes. There is no qualitative difference between these two cases.

Let $n$ be even. For any $s \in \Delta_m$ and active player $i$, let $L_i^s$ be the set of $m - \frac{n}{2}$ active players excluding $i$. Combined with $n - m$ deprived players and $i$, any $L \in L_i^s$ forms a minimum-winning coalition of size $\frac{n}{2} + 1$. Thus $L \in L_i^s$ is a set of $i$’s costliest coalition partners in $s$. Denote a generic element of $L_i^s$ by $L_{id}^s$, where $\phi = 1, \ldots, \lambda_i$ and $\lambda_i = |L_i^s| = \left\lceil \frac{m-1}{\frac{n}{2}} \right\rceil$.

Construct a sequential bargaining game $\Gamma(s)$ for each $s \in \Delta_m$ as follows: A set $I = \{1, \ldots, n\}$ of players bargain over a good of total value $\frac{1}{1-\delta}$ for period $\tau = 1, \ldots, \infty$. At the beginning of the game, one player is recognized as a proposer according to the recognition rule $s$ and proposes a division of the good $U \in \mathcal{U}$, where $\mathcal{U} = \{U \in \mathbb{R}^n | \sum_{i \in I} U_i = \frac{1}{1-\delta} \text{ and } U_i \geq 0, \forall i\}$. Observing $U$, all active players vote whether to accept or reject it, where active players refer to the players who have positive proposal power in $s$. If the proposal $U$ obtains $m - \frac{n}{2} + 1$ approval votes from active players, every player $i \in I$ receives one’s share $U_i$ and the game ends; otherwise the players receive $s_i$ for that period and the same procedure is resumed in the next period and repeated until an agreement is reached. The players discount the future by a same factor $\delta \in [0, 1)$. Thus if the game ends in period $k$ with an agreement $U$, each player’s payoff is $\sum_{\tau=1}^k \delta^{\tau-1} s_i + \delta^{k-1}(U_i - s_i)$.

I focus on symmetric stationary proposal and voting strategies. Player $i$’s proposal strategy is $\pi_i^s \in \Delta(\mathcal{U}) = \Pi_i^s$, where $\Delta(\mathcal{U})$ is the set of probability distributions over $\mathcal{U}$ and $\pi_i^s(U)$ is the probability that player $i$ proposes $U \in \mathcal{U}$. Assume that the support of $\pi_i^s$ is finite for all $i$. Player $i$’s voting strategy is $\alpha_i^s : \mathcal{U} \rightarrow \{\text{Accept, Reject}\}$. Let $\beta_i^s = (\pi_i^s, \alpha_i^s)$ be $i$’s strategy in the game $\Gamma(s)$ and $\beta^s = \times_{i \in I} \beta_i^s$ be a strategy profile. For any $U \in \mathcal{U}$ and $\beta^s$, let $I(U, \beta^s)$ be an indicator function where $I(U, \beta^s) = 1$ if $[\sigma : \alpha_i^s(U) = \text{Accept}] \geq m - \frac{n}{2} + 1$ and 0 otherwise.\(^5\)

I seek for no-delay symmetric stationary subgame perfect equilibria in stage-undominated voting strategies (SSPE) $\beta^{ss}$: for every proposer $i$ and non-proposer $j$,

\[
\pi_i^{ss}(U) > 0 \quad \Rightarrow \quad U \in \arg \max_{U' \in \mathcal{U}} \{I(U', \beta^{ss}) = 1\},
\]

\[
\alpha_i^{ss}(U) = \text{Accept} \quad \Leftrightarrow \quad U_i \geq \psi_i^{ss}(\beta^{ss})
\]

\(^5\)If $n$ is odd, $L_i^s$ contains $m - \frac{n+1}{2}$ other active players, and $\lambda_i = \left\lceil \frac{m-1}{\frac{n}{2}} \right\rceil$. Also $I(U, \beta^s) = 1$ if the number of players who accepts a proposal $U$ is $m - \frac{n}{2}$ and 0 otherwise.
where \( \psi_j(\beta^i) \) is player \( j \)'s reservation value in \( \beta^i \) defined as
\[
\psi_j^i(\beta^i) = s_i + \delta \left[ s_i \sum_{U: \pi_j^i(U) > 0} \pi_j^i(U)(I(U, \beta^i)U_i + (1 - I(U, \beta^i))\psi_j^i(\beta^i)) \right] + \sum_{j \neq i} s_j \sum_{U: \pi_j^i(U) > 0} \pi_j^i(U)(I(U, \beta^i)U_i + (1 - I(U, \beta^i))\psi_j^i(\beta^i)) \]. (13)

The state game \( \Gamma(s) \) is analogous to the Baron-Ferejohn model with voting rule \( q = m - \frac{n}{2} + 1 \) and recognition rule \( s \), and we have the following Lemma.

**Lemma 5.** For every \( s \in \Delta_m \), the state game \( \Gamma(s) \) has SSPE. In any equilibrium \( \beta^{sx} = (\pi^{sx}, \alpha^{sx}_i) \),
(i) for every voter \( j \in I \), \( a_j(U) = \text{Accept} \iff U_j \geq \psi_j^{sx} \);
(ii) for every proposer \( i \) and \( i \)'s proposal \( U^{i\phi} \) such that \( \pi_j^{sx}(U^{i\phi}) > 0 \),
\[
\begin{align*}
U_j^{i\phi} &= \begin{cases} 
\frac{1}{1 - \delta} - \sum_{j \in L_{i\phi}^j} \psi_j^{sx} & \forall j \in L_{i\phi}^j \\
\psi_j^{sx} & \forall k \not\in L_{i\phi}^j \cup \{ i \},
\end{cases} \\
U_i^{i\phi} &= \begin{cases} 
\psi_i^{sx} & \forall j \in L_{i\phi}^j \\
0 & \forall k \not\in L_{i\phi}^j \cup \{ i \},
\end{cases}
\end{align*}
\]
where \( L_{i\phi}^j = \arg \max_{L \in L_i} \left[ \frac{1}{1 - \delta} - \sum_{j \in L} \psi_j^{sx} \right] \); and
(iii) for every player \( i \), the equilibrium reservation value \( \psi_i^{sx} \) is given by
\[
\psi_i^{sx} = s_i + \delta \left[ s_i \sum_{U^{i\phi}: \pi_i^{sx}(U^{i\phi}) > 0} \pi_i^{sx}(U^{i\phi})U_i^{i\phi} + \sum_{j \neq i} s_j \sum_{U^{i\phi}: \pi_j^{sx}(U^{i\phi}) > 0} \pi_j^{sx}(U^{i\phi})U_i^{i\phi} \right].
\]

In SSPE of the state game \( \Gamma(s) \), each proposer forms a minimum winning and minimum-cost winning coalition and distributes the selected coalition partners their reservation values. All these coalition partners then accept the proposal since they are indifferent between accepting the proposal and rejecting it and waiting until the next period. Note that for any given \( \psi^s \in \Psi \) player \( i \) has at least one set of coalition partners \( L_{i\phi}^j \) that maximizes \( \frac{1}{1 - \delta} - \sum_{j \in L_{i\phi}^j} \psi_j^{s} \), and for any such \( L_{i\phi}^j \) there is a unique proposal \( U^{i\phi} \) that satisfies the coalition partners’ demands. It is also easy to see that in any SSPE \( \beta^{sx} \) of \( \Gamma(s) \), proposal \( U^{i\phi} \) that \( i \) proposes with a positive probability is associated with a unique set of costly coalition partners \( L_{i\phi}^s \). The existence of SSPE can be shown by constructing a non-empty, convex-valued and upper-hemicontinuous correspondence \( f = B \circ \psi^s : \Pi^s \to \Pi^s \) by \( f(\pi^s) = \{ \pi_i^s : \pi_i^s \in \Pi^s : \pi_i^s \in B(\psi^s(\pi^s)) \} \) and applying the fixed point theorem, where \( B : \Psi \to \Pi^s \) is a correspondence such that \( B_i(\psi^s) \) returns the set of \( i \)'s best response proposal strategies \( \pi_i^s \) given \( \psi^s \in \Psi \) and \( \psi^s(\pi^s) \) is a function that returns the players’ reservation values for each proposal strategy profile \( \pi^s \in \Pi^s \).

The next lemma is a key step in connecting each state game \( \Gamma(s) \) and its SSPE proposal strategy \( \pi^{sx} \) with the equilibrium proposal profile \( \mu(s) \) in the full dynamic game with endogenous proposal power.

**Lemma 6.** Let \( k \in \mathbb{N} \) and \( k \leq \frac{n}{2} + 1 \). For any \( U \in \mathcal{U} \) with \( k \) strictly positive entries, there exists \( y \in X \) such that \( U_i(y) = U_i \) for all \( i \), where \( U_i(y) \) is derived from \( \sigma(y) \) that constitutes an equilibrium.

Lemma 6 says that for any vector \( U \in \mathcal{U} \) with at most \( \frac{n}{2} + 1 \) strictly positive entries, there is an allocation \( y \in X \) that generates an equilibrium expected utility vector \( U(y) = U \). If \( k \leq \frac{n+1}{2} \), the corresponding \( y \) must be a status quo allocation in a state in \( \Delta_i \). By Lemma 2, we know that each player’s equilibrium expected utility \( U_i(y) \) from \( y \in \Delta_i \) is given by \( U_i(y) = \frac{m_i}{1 - \delta} \). Thus there is a unique corresponding allocation \( y \) for any
\( U \in \mathcal{U} \) with \( k \leq \frac{n+1}{2} \) strictly positive entries. If \( n \) is even and \( k = \frac{n}{2} + 1 \), the corresponding allocation \( y \) must be a status quo allocation in a state in \( \Delta_{n+1} \). In state \( y \) with \( \frac{n}{2} + 1 \) active players, each proposer chooses a single active player with the lowest expected utility as a costly coalition partner in equilibrium. Utilizing this equilibrium proposal behavior in \( y \) and setting \( U(y) = U \), we can obtain \( y \in \mathcal{X} \) as Example 2 demonstrates.

**Example 2.** If \( \delta = 0 \), the allocation \( y \) is trivially given by \( U = y \). Thus let \( \delta \in (0, 1) \).

(i) Let \( n = 6 \) and \( U = (\frac{4}{10(1-\delta)}, \frac{3}{10(1-\delta)}, \frac{2}{10(1-\delta)}, \frac{1}{10(1-\delta)}, 0, 0) \).

In equilibrium, players 1-3 guarantee the expected utility \( \frac{1}{10(1-\delta)} \) to player 4 and take the remainder when they are proposers and player 4 gives \( \frac{5}{10(1-\delta)} \) to player 3 and takes the remainder when she is a proposer in state \( y \). That is,

\[
U_1(y) = y_1 + \frac{9\delta y_1}{10(1-\delta)} = \frac{4}{10(1-\delta)}
\]
\[
U_2(y) = y_2 + \frac{9\delta y_2}{10(1-\delta)} = \frac{3}{10(1-\delta)}
\]
\[
U_3(y) = y_3 + \frac{9\delta y_3}{10(1-\delta)} + \frac{2\delta y_4}{10(1-\delta)} = \frac{2}{10(1-\delta)}
\]
\[
\text{and } U_4(y) = y_4 + \frac{8\delta y_4}{10(1-\delta)} + \frac{(1-y_4)\delta}{10(1-\delta)} = \frac{1}{10(1-\delta)}
\]

Solving above equations yields \( y = (\frac{4}{10-\delta}, \frac{3}{10-\delta}, \frac{2(10-4\delta+\delta^2)}{2(10-9\delta)}, \frac{2-\delta}{2(10-9\delta)}, \frac{1-\delta}{10-9\delta}, 0, 0) \), and \( \sum_{i=1}^6 y_i = 1 \). It is obvious that there is a unique \( y \in \mathcal{X} \) with \( U(y) = U \).

(ii) Let \( n = 6 \) and \( U = (\frac{5}{10(1-\delta)}, \frac{3}{10(1-\delta)}, \frac{2}{10(1-\delta)}, \frac{1}{10(1-\delta)}, 0, 0) \).

First consider \( y \in \mathcal{X} \) such that \( y_3 = y_4 \) and \( U(y) = U \). In equilibrium, players 1 and 2 guarantees the expected utility \( \frac{1}{10(1-\delta)} \) to player 3 or 4 each with probability \( \frac{1}{2} \) by symmetry, and player 3 and 4 choose each other and give \( \frac{1}{10(1-\delta)} \) when they are recognized to propose. That is,

\[
\text{for } i = 1, 2, \quad U_i(y) = y_i + \frac{9\delta y_i}{10(1-\delta)} = U_i
\]
\[
\text{for } j = 3, 4, \quad U_j(y) = y_j + \frac{9\delta y_j}{10(1-\delta)} + \frac{\frac{1}{2}(1-2y_j) + y_j}{10(1-\delta)} - \delta = y_j + \frac{(18y_j + 1)\delta}{20(1-\delta)} = U_j.
\]

Solving above equations, we have \( y = (\frac{5}{10-\delta}, \frac{3}{10-\delta}, \frac{2\delta}{2(10-9\delta)}, \frac{2-\delta}{2(10-9\delta)}, \frac{1-\delta}{10-9\delta}, 0, 0) \). Thus, we know that there exists \( y \in \mathcal{X} \) such that \( U(y) = U \).

However, such \( y \) is not unique because players 1 and 2 are indifferent between players 3 and 4 given the expected utility vector. Suppose that there is \( \bar{y} \) such that \( U(\bar{y}) = U \) and players 1, 2 and 3 selects player 4 as a coalition partner with probability 1 and players 4 selects player 3 with probability 1 in state \( y \). Then,

\[
\text{for } i = 1, 2 \quad U_i(\bar{y}) = \bar{y}_i + \frac{9\delta \bar{y}_i}{10(1-\delta)} = U_i
\]
\[
U_3(\bar{y}) = \bar{y}_3 + \frac{9\delta \bar{y}_3}{10(1-\delta)} + \frac{\delta \bar{y}_4}{10(1-\delta)} = \frac{1}{10(1-\delta)}
\]
\[
U_4(\bar{y}) = \bar{y}_4 + \frac{(9\bar{y}_4 + 1 - \bar{y}_4)\delta}{10(1-\delta)} = \bar{y}_4 + \frac{(1 + 8\bar{y}_4)\delta}{10(1-\delta)} = \frac{1}{10(1-\delta)}
\]

Solving above equation, we have \( \bar{y} = (\frac{5}{10-\delta}, \frac{3}{10-\delta}, \frac{4-\delta}{2(10-9\delta)}, \frac{2\delta}{2(10-9\delta)}, \frac{1-\delta}{10-9\delta}, 0, 0) \). Thus we have at least two distinct \( y, \bar{y} \in \mathcal{X} \) such that \( U(y) = U(\bar{y}) = U \) for any \( \delta \in (0, 1) \).
In almost all cases, the allocation \( y \in X \) such that \( U(y) = U \) is unique for any \( U \in \mathcal{U} \) with \( k \leq \frac{n}{2} + 1 \) positive entries in equilibrium. If \( n \) is odd; or if \( n \) is even and \( k \leq \frac{n}{2} \), the allocation \( y \) is unique as a direct consequence of Lemma 2. There are multiple allocations that satisfy \( U(y) = U \) if \( n \) is even, \( k = \frac{n}{2} + 1 \) and either when there are more than one player having the lowest positive expected utilities or when there is a single player having the lowest expected utility but more than one players’ expected utilities are the second lowest in \( U \). When there are more than one players with the lowest expected utilities as in Example 2, all other players are indifferent in choosing any of these players as a costly coalition partner and are allowed to mix insofar as the mixing probabilities generates the given expected utility vector. Suppose that player 3 and 4’s expected utilities given in \( U \) are the lowest and the same. In equilibrium, player 3 and 4’s payoffs from their own proposals are the same since they choose each other as a coalition partner and take the remainder expected utilities. Their equilibrium expected utility difference, if any, then stems from their recognition probabilities and the probabilities to be chosen by other players as a coalition partner. If other players collectively choose player 4 with a higher probability than player 3, player 4’s current share needs to be smaller than player 3’s to make their expected utilities equal. As one can easily see, there are infinitely many possible combinations of other players’ mixing probabilities and player 3 and 4’s shares \( y_3 \) and \( y_4 \) that make their expected utilities the same. Although there are multiple allocations \( y \in X \) such that \( U(y) = U \) for \( U \in \mathcal{U} \) with \( \frac{n}{2} + 1 \) positive entries in the case \( n \) is even, the converse is not true. Lemma 7 in Appendix shows that the equilibrium expected utility vector from any \( y \) with \( \frac{n}{2} + 1 \) active players is unique in all equilibria.

Proposition 2 establishes existence of symmetric stationary Markov Perfect Nash equilibria in stage-undominated voting strategies in the entire game by connecting the SSPE of the state game \( \Gamma(s) \) and the equilibrium.

**Proposition 2.** For any \( \delta \in [0, 1) \), there exist equilibria. In any equilibrium \( \sigma \) and \( \tilde{\sigma} \), \( U^\sigma(s) = U^\tilde{\sigma}(s) \) for each \( s \in S \) and all \( \delta \in [0, 1) \).

In all states in \( \Delta \), every active player has a unique optimal proposal assigning the entire dollar to oneself. These proposals are accepted by all deprived players who constitute a decisive coalition together with the proposer. Further the equilibrium expected utility vector \( \mathbb{E}(s) \) is unique for each \( s \in \Delta \). Thus we only need to focus on the states in \( \Delta_m \). Consider the state game \( \Gamma(s) \) for each \( s \in \Delta_m \). In any SSPE of \( \Gamma(s) \), each active player’s equilibrium proposal plan is to propose \( U^i \) with probability \( \pi \sigma(s) \), which generate the SSPE reservation value vector \( \psi \). When the number of players \( n \) is odd, any proposal \( U^i \) player \( i \) uses with positive probability in a SSPE has exactly \( m - \frac{n-1}{2} \) positive entries since each player chooses the minimum number of costly coalition partners \( m - \frac{n-1}{2} \). Since \( m \leq n \), \( U^i \) has at most \( \frac{n+1}{2} \) positive entries. Then Lemma 6 implies that there is a unique corresponding allocation \( y^i \in X \) for each \( U^i \) in the support of \( \pi \sigma(s) \) such that \( U^i = U(y^i) \). Now turning back to the entire game, let each player’s proposal plan be \( \mu_i(y^i) = \pi \sigma(s) \) for each \( s \in \Delta_m \) and the acceptance sets be \( y \in A_i, s \Rightarrow U_i(y) \geq U_i(s) \). Writing each player’s expected utility from \( s \), we have

\[
U_i(s) = s_i + \delta[s_i \sum_{y^i \in \mathbb{R}_+} \mu_i(y^i)U_i(y^i) + \sum_{j \neq i} s_j \sum_{y^j \in \mathbb{R}_+} \mu_j(y^j)U_j(y^j)] = \psi \sigma(s)
\]

by \( \mu_i(y^i) = \pi \sigma(s) \) and \( U_i = U_i(y^i) \) for every \( i \). Obviously, every proposer’s proposal plan is optimal in \( s \) since each \( y^i \) in the support of \( \mu_i(s) \) involves with a minimum cost winning coalition partners by construction of the SSPE of \( \Gamma(s) \). The proof for the \( n \) even case is essentially the same except that the allocation \( y^i \) that corresponds to each \( U^i \) is not unique if the number of active players in the initial state is \( m = n \). Let \( \{y^i_{\eta \in \mathbb{N}}\}_{\eta = 1} \) be any finite selection of \( y \) such that \( U(y) = U^i \). As long as \( \sum_{\eta = 1} \mu_i(y^i_{\eta \in \mathbb{N}}) = \pi \sigma(s) \), any arbitrary finite selection of \( \{y^i_{\eta \in \mathbb{N}}\}_{\eta = 1} \) does not affect \( U(s) \). Let \( \mu_i \) be the collection of thus-constructed \( \mu_i(s)_{s \in S} \) and \( A_i \) be the collection of \( A_i(s)_{s \in S} \). Then it is easy to see that \( \sigma = ((\mu_i, A_i))_{i \in \mathbb{N}} \) is an equilibrium in which every player’s proposals are optimal and every players’ voting strategies are stage-undominated.
in all states \( s \in S \). Given that \( U(s) = \psi^* \), the uniqueness of equilibrium expected utility vector \( U(s) \) for each \( s \in S \) follows from the uniqueness of the SSPE payoffs \( \psi^* \) proven in Eraslan (2002) and Eraslan and McLennan (2013), and the proof is omitted.

5 Conclusion

I have presented and analyzed an infinite-horizon divide-the-dollar game in which the players’ resources obtained in the current period determine their proposal power in the next period. Although the players possess voting rights collectively to prevent further concentration of proposal power, such dynamics do not occur. Once a player is deprived of resources and proposal power, that player approves every proposal because there is no hope to receive any portion of the dollar and regain proposal power in the future. The proposers in each period take advantage of this situation. Once there are enough players who are willing to vote for everything, any recognized proposer immediately becomes a tyrant. Even when the proposer is not able to directly propose her tyranny, her interest lies in eliminating as many players as possible, especially the players who have a greater portion of the resources and better prospects in the future competition toward tyranny. Hence, proposers in these states form minimum-winning and minimum-cost winning coalitions. The convergence to tyranny occurs at most within three rounds regardless of the number of players, discount factors and distribution of resources and proposal power in the initial states.

In reality, the deprived players can be understood as the individuals or groups who do not have enough resources to make their voice heard in political arena. The model shows that the increasing influence of the resource-rich elites in the economic and political sphere we witness today might be detrimental for the health and legitimacy of democracy. In the presence of the resource-power interdependency, the deprived members of a society remain deprived and become indifferent and uninterested in the political competition in which the resource-rich elites dominate political agenda. The distrust and indifference of these deprived players in the end enable an extreme degree of domination. The result resonates with the increasing political indifference of voters evidenced by decreasing turnout in recent elections and increasing economic inequality, and raises concerns about long-run consequences of such trends.

The resources in this model can be interpreted as political as well. Interpreting the resources as political is relevant in situations where political elites enjoy greater freedom in structuring the rules for future political competition. Political groups or parties that control a larger number of seats in constitutional assemblies may use this power to shape the rules pertaining to who can run for offices, who can vote, how to translate votes to seats, to what extent to control the media, and who control the budget process and military, to list a few. These institutional arrangements are the resources necessary to reproduce political power in the future legislatures, and the members in the legislature in turn have the authority to revise existing institutional arrangements. We have observed numerous examples in history and contemporary politics of elites who, having political power, shape political institutions in order to keep hold of their power. For instance, white primaries, which barred African Americans from participating in primary elections, were codified in party rules and implemented in many Southern Democratic parties until the mid-20th century, allowing party leaders to control the elections (Weeks, 1948). Upon seizing the executive power, Mikheil Saakashvili, the former president of Georgia who led the Rose Revolution in 2003, strengthened the presidential power and controlled the police, media, legislature and judiciary through a series of constitutional amendments that allowed him to maintain power (Schofield et al., 2012; Tudoroiu, 2007). Conceptualizing the resources in this model as political would help understand the relationship between institutional developments and configuration of power in new democracies.

The result of inevitable domination of a single group relies on a set of modeling assumptions. Examining the relevance of the key assumptions and extending the model will help us to identify the conditions under which one, a few, or many members of a society share the resources and agenda-setting power in the long
run. First, I have assumed in the model that the players are risk neutral. But risk-averse and farsighted players may have incentives to cooperate and compromise as in Bowen and Zahran (2012). As an example, suppose that there are four players and the players short-term utility is \( u_i(x) = u(x_i) \), \( u' > 0, u'' < 0 \) and \( u'(0) = \infty \). In this setting, the utility loss from being deprived cannot be compensated by any other provisions. Thus each player tries to minimize the probability of being deprived, and it enables the equal allocations of the dollar among any three players to emerge in equilibrium as long-run outcomes. To see this, consider the following strategy profile: In the states with at least two deprived players, all active players propose one’s tyranny; in the states in which three players equally share the dollar, all active players propose the status quo allocation; and in all other states all active players propose equal allocations of the dollar among three players by randomly selecting two other players so that every player has a positive chance to be deprived. Except for the tyrannical states and the states in which three players equally share the resources and power, the active players in all other states are vulnerable to the possibility of future deprivation. Thus the absorbing state in this setting will be either tyranny or equal sharing of wealth and power among three players, depending on the initial state in an equilibrium. A fuller examination on the role of risk-aversion beyond this extreme and simple example will shed light on the size and forms of long-run stable coalitions.

Another extension of the model involves the assumption of perfect correlation between resources and power. The resource-rich elites may be bounded by existing institutional rules such as caps on campaign contributions, public finance for elections, or rules assigning a specific amount of time for media exposure to each candidate. One can modify the model and study the long-run effects of these rules by setting upper or lower limits on the players’ proposal power. Permanent tyranny obviously cannot be supported in equilibrium in the presence of unalienable default proposal rights, however small it is. Incorporating uncertainty in the inter-period resource-power conversion will also help model the situations in which elites cannot precisely predict their political influence due to the complex interplay of other factors. Although the full analyses of these extensions are beyond the scope of the paper, the primary contribution and merit of the model presented here is its tractability that enables various extensions to deepen our understanding of the extent and limits of resource and power interdependency.
References


Appendix: Formal proofs

Proof of Proposition 1. For the proposals and acceptance sets in $s$ given in the Proposition 1, all players accept other players’ proposals if and only if the proposal is weakly better than the status quo. Hence, every player’s voting strategy is stage-undominated. Also, no player has an incentive to deviate from this voting strategy since rejecting a proposal $y$ such that $U_j(y) \geq U_j(s)$ gives either the same or lower level of expected utility to $j$.

Given these voting strategies of other players, proposer $i$ selects some sets of decisive coalition partners $L_{i\phi} \in \vec{L}_i$ and proposes allocation $y^{i\phi} \in X$ that gives the current expected utilities to its coalition partners in $L_{i\phi}$ and nothing to all other players, and takes the remainder expected utility by himself. Proposer $i$ is worse off if $i$ proposes another allocation $y$ that gives more than the current expected utilities to some of his coalitions partners, since such a proposal will be passed and the proposer’s payoff from the proposal is the remainder.

Proposer $i$’s problem is to choose a set $L_{i\phi}$ of decisive coalition partners whose sum of expected utilities is the lowest among all sets of decisive coalition partners as given in (10). That is, proposer $i$ proposes $y^{i\phi}$ associated with the set of coalition partners $L_{i\phi}$ with positive probability only if $L_{i\phi}$ solves the problem $\min_{L \in \vec{L}_i} \sum_{j \neq i} U_j(s)$. Evidently, proposer $i$ cannot improve its payoff from his proposal by proposing $y$ that is associated with $L_{i\bar{\phi}}$ that is not a solution to the minimization problem. Proposer $i$ is indifferent between any distinct $L_{i\phi}$ and $L_{i\phi'}$ that satisfies (10) and their associated proposals $y^{i\phi}$ and $y^{i\phi'}$ for any $\eta, \eta'$, but in equilibrium the mixing probability must be consistent with the expected utility from $s$ given in (9). It is straightforward to see that the strategy profile described above induces the expected utility for every $i$ and every $s$.

Proof of Lemma 1. First I show that for any $s \in S$, $s_i = 0$ if and only if $U_i(s) = 0$ in equilibrium. The sufficiency proof is trivial since $U_i(s) = s_i + \delta v_i(s)$, and both $s_i$ and $v_i(s)$ are non-negative. Hence, $U_i(s) = 0$ implies $s_i = 0$. For necessity, suppose on the contrary that $s_i = 0$ but $U_i(s) > 0$. If $\delta = 0$, we already have a contradiction by $U_i(s) = s_i = 0$. Thus, assume that $\delta > 0$. Since $U_i(s) = s_i + \delta v_i(s)$ and $s_i = 0$, it must be true that $v_i(s) > 0$. By $s_i = 0$, player $i$ has no proposal power. Hence, $v_i(s) > 0$ implies that there exists at least one active player who allocates positive share to $i$ when recognized. By Proposition 1, every proposer guarantees exactly the currently expected utilities to its coalition partners and no more in equilibrium. Let $\alpha > 0$ be the probability that $i$ is chosen as a coalition partner by any active player and receives $U_i(s)$. Then, $U_i(s) = \delta v_i U_i(s)$, and we have $U_i(s) = 0$ for any $\delta, \alpha > 0$, a contradiction. The second part of the Lemma directly follows from the above discussion and the fact that $U_i(s) \geq 0$ for any $s \in X$.

Proof of Lemma 2. (i) If $n$ is even, there are $n - l \geq \frac{n}{2}$ deprived players in $s$ who collectively constitute a decisive coalition together with $i$, and deprived players accept all proposals. Given that, player $i$ proposes the entire dollar to oneself. If $n$ is odd, $n - l \geq \frac{n-1}{2}$ and same argument applies.

(ii) In $s$, each active player is recognized with probability $s_i$ and takes the entire dollar for all subsequent periods, which gives an expected utility $U_i(s) = s_i + \delta v_i(s_i \frac{1}{1-\delta}) = \frac{s_i}{1-\delta}$ for deprived players, $s_i = U_i(s) = 0$.

Proof of Lemma 4. If $s_j = 0$, the statement directly follows from Lemma 1. If $s_i = s_j$, the statement also directly follows by symmetry. So, assume that $x_i \geq x_j > 0$. For $s \in \Delta_i$, it is obvious from Lemma 2. So, assume that $s \in \Delta_m$, where $\frac{m}{2+1} < m \leq n$. I prove for the case of odd $n$. The proof for even $n$ is the same if replacing $\frac{m+1}{2}$ below with $\frac{m}{2}$.

I suppose a contradiction that $s_j > s_j$ but $U_i(s) < U_j(s)$ and show that there is a contradiction if $U_i(s)$ and $U_j(s)$ are derived from an equilibrium. Align the players so that $U_k(s) \geq U_{k+1}(s)$ for all $k = 1, \ldots, n-1$. Since $U_i(s) < U_j(s)$, now we have $i > j$. There are three cases to consider: (i) there are more than $\frac{n+1}{2}$ players whose index is below $i$ ($j < i < \frac{n+1}{2}$), which implies that both $i$ and $j$ do not receive any share of the dollar in other players’ proposals in equilibrium; (ii) there are less than $\frac{n-1}{2}$ players whose index is below
proposals; and (iii) \( i \geq \frac{n+1}{2} \) and \( j \leq \frac{n+1}{2} \) with \( i > j \).

(i) If there are more than \( \frac{n+1}{2} \) other players, including deprived players, whose expected utilities are strictly less than \( U_i(s) \) and \( U_j(s) \), both players are not included in other players’ coalitions. Further, both \( i \) and \( j \)'s proposals yield the same expected utility since they form a decisive coalition with either the same set of other players or different set of players whose sum of expected utilities are the same and the lowest. Let \( \bar{U} \) denote \( i \) and \( j \)'s expected utility from their proposals. Since no other players’ assign them any expected utilities (or any positive share of the dollar) in their proposals, we write \( U_i(s) \) and \( U_j(s) \) as

\[
U_i(s) = s_i + \delta s_i \bar{U} \quad \text{and} \quad U_j(s) = s_j + \delta s_j \bar{U}
\]

Since \( s_i > s_j \), we have \( U_i(s) > U_j(s) \), a contradiction.

(ii) If there are at most \( \frac{n-3}{2} \) players, including \( i \), whose expected utilities are strictly less than \( U_j(s) \), every other player guarantees the current expected utilities to both \( i \) and \( j \) in one’s proposal since \( i \) and \( j \) are one of the \( \frac{n+1}{2} \) least expensive coalition partners. Let \( \bar{U} \) denote the sum of expected utilities of the \( \frac{n-3}{2} \) players whose expected utilities are lowest excluding player \( i \) and \( j \). Player \( i \)'s proposal in equilibrium should guarantee \( \bar{U} + U_j(s) \) to other players in order to obtain a majority support, and player \( j \) should pay \( \bar{U} + U_j(s) \). For player \( i \), other players are recognized with probability \( 1 - s_i \) and gives player \( i \) the expected utility of \( U_j(s) \); and for player \( j \), the probability is \( 1 - s_j \) and \( j \) receives \( U_j(s) \). Thus

\[
U_i(s) = s_i + \delta s_i \left( \frac{1}{1 - \delta} - \bar{U} - U_j(s) \right) + (1 - s_i)U_j(s)
\]

\[
= s_i + \delta s_i \left( \frac{1}{1 - \delta} - \bar{U} - U_j(s) - U_j(s) + U_j(s) \right)
\]

\[
= s_i + \delta s_i \left( \frac{1}{1 - \delta} - \bar{U} - U_j(s) - U_j(s) \right)
\]

Similarly,

\[
U_j(s) = \frac{s_j + \delta s_j \left( \frac{1}{1 - \delta} - \bar{U} - U_j(s) - U_j(s) \right)}{1 - \delta}
\]

and comparing \( U_i(s) \) and \( U_j(s) \) above, we see that \( U_i(s) > U_j(s) \) by \( s_i > s_j \), a contradiction.

(iii) Simply the case in which \( i \geq \frac{n+1}{2} \) and \( j \leq \frac{n+1}{2} \) corresponds to all cases other than (i) and (ii).

First, consider each player’s expected utilities from one’s own proposal. Let \( \bar{U} \) be the sum of the expected utilities of the first \( \frac{n+1}{2} \) players. Remember that \( U_k(s) \geq U_{k+1}(s) \) for all \( k = 1, \ldots, n-1 \). Regardless of player \( i \)'s index given that \( i \geq \frac{n+1}{2} \), players \( i \) can obtain \( \bar{U} + U_i(s) \) if he is recognized to propose. To see this, notice that the sum of all players’ expected utilities from any division is \( \frac{1}{1 - \delta} \), and

\[
\frac{1}{1 - \delta} = \sum_{k=1}^{\frac{n}{2}} U_k(s) + \sum_{k=\frac{n}{2}+1}^{n} U_k(s) = \bar{U} + \sum_{k=\frac{n}{2}+1}^{n} U_k(s) + U_i(s)
\]

Since player \( i \) in his proposal guarantees the current expected utilities to \( \frac{n-1}{2} \) other players with lowest expected utilities and takes the remainder, \( i \)'s expected utility from his own proposal is

\[
\frac{1}{1 - \delta} - \sum_{k=\frac{n}{2}+1, k\neq i}^{n} U_k(s) = \bar{U} + U_i(s)
\]

Similarly, player \( j \) in his proposal gives the current expected utilities to the \( \frac{n-1}{2} \) players with the lowest expected utilities and takes the remainder. Thus player \( j \)'s expected utility from his own proposal is

\[
\frac{1}{1 - \delta} - \sum_{k=\frac{n}{2}+1}^{n} U_k(s) = \bar{U} + U_j(s)
\]

Thus player \( j \)'s expected utility from his own proposal is

\[
\frac{1}{1 - \delta} - \sum_{k=\frac{n}{2}+1}^{n} U_k(s) = \bar{U} + U_j(s)
\]

Denote \( \bar{U} = \bar{U} - U_j(s) + U_{\frac{n}{2}+1}(s) \), then player \( j \)'s expected utility from his own proposal is \( \bar{U} + U_j(s) \). By \( j \leq \frac{n+1}{2} \), we have \( U_j(s) \geq U_{\frac{n+1}{2}}(s) \). Accordingly, \( \bar{U} \leq \bar{U} \).

Now let \( \alpha_i \) and \( \alpha_j \) each denote the probability that player \( i \) and \( j \) are chosen as a coalition partner in other players’ proposals. First notice that \( \alpha_i > 0 \) since \( i \leq \frac{n+1}{2} \) and \( s \in \Delta_m \). Next if \( \alpha_j > 0 \), then \( \alpha_i = 1 - s_j \). Given that \( j \leq \frac{n+1}{2} \), \( \alpha_j > 0 \) occurs only when \( j = \frac{n+1}{2} \) or when \( U_j(s) = U_{\frac{n+1}{2}}(s) \). By \( i > j \), then, \( i \geq \frac{n+1}{2} \) and \( i \)

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must be included in all other players’ coalitions, implying \( \alpha_i = 1 - s_i \). In this case, \( \alpha_j < 1 - s_j \) since there are at least \( \frac{n - 1}{2} \) other players whose expected utilities are strictly less than \( U_j(s) \) and all players \( k < j \) do not include \( j \) in their coalitions. In sum, we have either \( \alpha_j = 0 \) and \( \alpha_i > 0 \) or \( \alpha_j > 0 \) and \( \alpha_i = 1 - s_i \). Note that \( s_i + \alpha_i > s_j + \alpha_j \) in any case.

Using above, we write \( U_i(s) \) and \( U_j(s) \):

\[
U_i(s) = s_i + \delta[s_i(\bar{U} + U_i(s)) + \alpha_i U_i(s)] = s_i + \delta s_i \bar{U} + \delta(s_i + \alpha_i)U_i(s)
\]

and

\[
U_j(s) = \frac{s_j(1 + \delta \bar{U})}{1 - \delta(s_j + \alpha_j)} < \frac{s_j(1 + \delta \bar{U})}{1 - \delta(s_i + \alpha_i)} \quad \text{by} \ s_i + \alpha_i > s_j + \alpha_j
\]

\[
< \frac{s_j(1 + \delta \bar{U})}{1 - \delta(s_i + \alpha_i)} = U_j(s) \quad \text{by} \ s_i > s_j \text{and} \ \bar{U} \geq \bar{U}
\]

Thus, we have a contradiction that \( U_i(s) > U_j(s) \). Since (i), (ii), and (iii) cover all possible cases, it completes the proof. ■

**Proof of Lemma 5.** Statements (i)-(iii) are straightforward. Let \( \psi^{s*} \) be a vector of equilibrium reservation values in \( \Gamma(s) \). Every player \( j \in I \) votes for proposal \( V \) if and only if it gives at least one’s reservation value \( \psi_j^{s*} \), which leaves \( j \) no incentive to deviate from the voting strategy in (i). By Lemma 1, \( n - m \) deprived players vote for any proposal. Knowing other players’ voting strategies, hence, proposer \( i \) needs to buy at least \( m - \frac{n}{2} \) other active players’ votes by giving them at least their reservation values. Since \( i \) takes the remainder after paying its coalition partners’ demands, \( i \) should choose exactly \( m - \frac{n}{2} \) other costly coalition partners and give their reservation values but no more. Thus, any proposal \( U_j^0 \) of player \( i \) gives \( \psi^{s*}_j \) to all \( j \in L_{i}^0 \) for some \( L_{i}^0 \in L_i^s \), and \( i \)’s problem is to find \( L_{i}^0 \in \arg\max_{L_{i} \in L_i^s} \left[ \frac{1}{1 - \delta} - \sum_{j \in L_i^0} \psi_j^{s*} \right] \) as given in (ii). It is obvious that \( U_j^0 \) that guarantees the demands of \( L_{i}^0 \) is unique for each \( L_{i}^0 \), and the converse is also true given \( \psi^{s*} \). For any \( U_j^0 \) and \( U_j^{0*} \) associated with the set of costly coalition partners \( L_{i}^0 \) and \( L_{i}^0 \), that solve the maximization problem in (ii), proposer \( i \) is indifferent, but the mixing probability \( \pi_i^{s*} \) of these proposals must induce \( \psi^{s*} \) given in (iii). To see that the reservation value \( \psi_i^{s*} \) is induced by \( \pi^{s*} \) and \( \alpha^i \) for all \( i \), notice that player \( i \) receives \( s_i \) for that period in case of rejection, and the bargaining resumes in the next period. In the next period, \( i \) is recognized as a proposer with probability \( s_i \) and proposes each \( U_j^0 \) with probability \( \pi_i^{s*}(U_j^0) \) and receives \( U_j^0 \). With probability \( s_j \), another player \( j \) is selected as a proposer and proposes each \( U_j^0 \) with probability \( \pi_j^{s*}(U_j^0) \) which again gives player \( i \) a value \( U_i^0 \).

To prove the existence of equilibrium \( \beta^{s*} \) of \( \Gamma(s) \), let \( \hat{\pi}_i^s(L_{i}^0) = \pi_i(U_j^0) \), where \( \hat{\pi}_i^s(L_{i}^0) \) is the probability player \( i \) chooses \( L_{i}^0 \in L_i^s \) and proposes \( U_j^0 \) as given in (ii). Denote the space of \( \hat{\pi}_i^s \) by \( \hat{\Pi}_i^s \), where \( \hat{\Pi}_i^s \) is a \( \lambda^s - 1 \) dimensional unit simplex, and let \( \hat{\pi}^s = \chi_{i \in I} \hat{\pi}_i^s \) and \( \hat{\Pi}^s = \chi_{i \in I} \hat{\Pi}_i^s \). Then can rewrite each \( i \)’s reservation value as a function of \( \hat{\pi}^s \) in \( \hat{\Pi}^s \). That is, define \( \psi_i^s : \hat{\Pi}^s \to \mathbb{R} \) by

\[
\psi_i^s(\hat{\pi}^s) = s_i + \delta \left[ \sum_{\phi=1}^{L_{i}^0} \hat{\pi}_i^s(L_{i}^0) \left( \frac{1}{1 - \delta} - \sum_{j \in L_{i}^0} \psi_j^s(\hat{\pi}^s) \right) + \sum_{j \neq i} \sum_{\phi=1}^{L_{i}^0} \hat{\pi}_i^s(L_{i}^0) \psi_j^s(\hat{\pi}^s) I_{L_{i}^0}(i) \right],
\]

where \( I_{L_{i}^0}(i) = 1 \) if \( i \in L_{i}^0 \) and 0 otherwise. Let \( \psi^s = \chi_{i \in I} \psi_i^s \). Note that \( \psi^s \in \mathcal{U} \), \( \psi^s \) is continuous in \( \pi^s \), and \( \pi^s \) uniquely determines \( \psi^s(\pi^s) \).
A SSPE $\beta^{ss} = (\pi^{ss}, a^{ss})$ exists if and only if there exists $\hat{\pi}^{ss} = \pi^{ss}$ such that, for every $i$,

$$\hat{\pi}^{ss}(L_{i|\theta}) > 0 \Rightarrow L_{i|\theta} \in \arg \max_{l \in L_i} \left\{ \frac{1}{1 - \delta} - \sum_{j \in L} \psi_j \right\},$$

(14)
given $\psi^{ss}$ and $\psi^{ss} = \psi(\hat{\pi}^{ss})$.

For any $\psi \in \mathcal{U}$, let $B_i(\psi)$ be the set of $\hat{\pi}_i \in \hat{\Pi}_i$ that satisfies (14). That is $B_i : \mathcal{U} \Rightarrow \hat{\Pi}_i$ is player $i$’s best response correspondence for any vector of reservation values. Let $\hat{\mathcal{L}}_i(\psi)$ be the set of $L \in \mathcal{L}_i$ that maximizes $\frac{1}{1 - \delta} - \sum_{j \in L} \psi_j$ given $\psi$. For any $\psi$, $\hat{\mathcal{L}}_i(\psi)$ is non-empty; hence $B_i(\psi)$ is non-empty for any $\psi$. Also $B_i(\psi)$ is the set of all probability distributions over $\hat{\mathcal{L}}_i(\psi)$ since $i$ is indifferent between all sets in $\hat{\mathcal{L}}_i(\psi)$; hence, $B_i(\psi)$ is compact and convex-valued for all $\psi$. By the Theorem of the Maximum, $B_i$ is upper hemi-continuous. Let $B = \times_{i \in \mathcal{I}} B_i$. Then $B$ is also non-empty, compact- and convex-valued and upper hemi-continuous. Define a correspondence $f = B \circ \psi : \hat{\Pi} \Rightarrow \hat{\Pi}^o$ by $f(\hat{\pi}) = \{ \hat{\pi}' \in \hat{\Pi}^o | \hat{\pi}' \in B(\psi(\hat{\pi}')) \}$. Then, $f$ is upper hemi-continuous, and $f(\hat{\pi})$ is non-empty and convex-valued for all $\hat{\pi}$. By Kakutani’s fixed point theorem, there exists a fixed point $\hat{\pi}^{ss}$ such that $\psi^{ss} = \psi(\hat{\pi}^{ss})$ and $\hat{\pi}^{ss} \in B(\psi^{ss})$. Obviously, all $L_{i|\theta}$ such that $\hat{\pi}^{ss}(L_{i|\theta}) > 0$ solves (14) for every $i$ given $\psi^{ss}$, and $\psi^{ss} = \psi(\hat{\pi}^{ss})$. Setting $\pi^{ss} = \hat{\pi}^{ss}$, then, $\beta^{ss}$ is a SSPE of the game $\Gamma(\pi,s)$.

**Proof of Lemma 6.** (i) Odd $n$: Consider $U \in \mathcal{U}$ with $k$ strictly positive entries and $n - k$ zero entries. Let $y \in X$ be such that $y_i = (1 - \delta)U_i$ for all $i$. Note that $\sum_i y_i = 1$ and $|\{i \in I | y_i > 0\}| = k$. Thus, $y, \Delta_k$ and $y \in \Delta$, where $1 \leq l \leq \frac{n + 1}{2}$. By Lemma 2, we know that $U_i(y) = \frac{y_i}{(1 - \delta)} = U_i$ if players’ proposals and acceptance sets in $y$ satisfy the equilibrium conditions.

(ii) Even $n$ with $k \leq \frac{n}{2}$: The proof is identical with (i) and there is $y \in X$ such that $y_i = (1 - \delta)U_i$ for all $i$.

(iii) Even $n$ with $k = \frac{n}{2} + 1$: Consider $U \in \mathcal{U}$ with $k$ strictly positive entries and $n - k$ zero entries and let $U_i \geq U_{i+1}$ for all $i = 1, \ldots, n - 1$. We seek for $y$ such that $U_i(y) = U_i$ for all $i$. From Lemma 1, $y \in \Delta_{\frac{n}{2} + 1}$ in equilibrium since $U_i(y) = 0$ if and only if $y_i = 0$. From Proposition 1 and Lemma 3, we know that every proposer forms minimum-cost winning coalitions and chooses minimum number of costly coalition partners in $y \in \Delta_{\frac{n}{2} + 1}$. Thus, every active player chooses $(\frac{n}{2} - 1) - \frac{n}{2} = 1$ other active player with the lowest expected utility as a coalition partner in $y$ in equilibrium. If $U_{\frac{n}{2} + 1} < U_{\frac{n}{2}} < U_i$ for all $i < \frac{n}{2}$, we obtain $y$ by solving the following equations:

$$U_{\frac{n}{2} + 1}(y) = U_{\frac{n}{2} + 1} = y_{\frac{n}{2} + 1} + \delta(y_{\frac{n}{2} + 1}(\frac{1}{1 - \delta} - U_{\frac{n}{2}}) + (1 - y_{\frac{n}{2} + 1})U_{\frac{n}{2} + 1})$$

(15)

$$U_{\frac{n}{2}}(y) = U_{\frac{n}{2}} = y_{\frac{n}{2}} + \delta(y_{\frac{n}{2}}(\frac{1}{1 - \delta} - U_{\frac{n}{2} + 1}) + y_{\frac{n}{2} + 1}U_{\frac{n}{2}})$$

(16)

for all $i \neq \frac{n}{2}, \frac{n}{2} + 1$, $U_i(y) = U_i = y_i + \delta(y_i(\frac{1}{1 - \delta} - U_{\frac{n}{2} + 1}))$.

and the solution is

$$y_{\frac{n}{2} + 1} = \frac{U_{\frac{n}{2} + 1}(1 - \delta)^2}{1 - \delta(1 - \delta)(U_{\frac{n}{2} + 1} + U_{\frac{n}{2}})}$$

$$y_{\frac{n}{2}} = \frac{U_{\frac{n}{2}}(1 - \delta y_{\frac{n}{2} + 1})(1 - \delta)}{1 - \delta(1 - \delta)U_{\frac{n}{2} + 1}}$$

(17)

for all $i \neq \frac{n}{2}, \frac{n}{2} + 1$, $y_i = \frac{U_i(1 - \delta)}{1 - \delta(1 - \delta)U_{\frac{n}{2} + 1}}$.

If there are $r$ players whose expected utilities are the same and the lowest, solving equation (15) by replacing
(1 - y^2_{k+1}) with \frac{1}{\delta} and U_2^\delta with U_2^{y_{k+1}} gives y_{k+1}^2 for all r players with the lowest expected utilities. All other players’ shares y_i are given by equation (17). Finally, if there are r players with the second lowest expected values, i.e., |\{j : U_j = U_2^{y_{k+1}}\}| = r, replacing y_{k+1}^2 with \frac{y_{k+1}^2 - y_2^2}{1 - \delta} and solving equation (16) gives y_2 for the r players and other players’ shares are as given in (15) and (16). It is tedious but straightforward to check that thus obtained y satisfies \sum_{i=1}^{2} y_i = 1, meaning y \in X. It is enough to show that there exists at least one y \in X such that U(y) = U, and it completes the proof.

Lemma 7. For each \delta \in (0, 1) and y \in \Delta^2_{k+1}, U(y) is unique in all equilibria.

Proof of Lemma 7. If \delta = 0, U(y) = y. Thus suppose \delta \in (0, 1). Consider any y \in \Delta^2_{k+1} and align the players so that y_1 \geq \ldots \geq y^2_{k+1} > 0 = \ldots = y_n. By Lemma 4 we know that U_i(y) \geq U_1(y) for i = 1, \ldots n - 1 in equilibrium. Let \sigma(y) and \hat{\sigma}(y) be two distinct equilibrium proposals and acceptance sets in y, and denote the expected utility vectors from these by U(y) and U^\sigma(y), respectively. I show that U(y) = U^\sigma(y). Suppose not. Then there exist at least two players i, j \leq \frac{2}{\delta} + 1 such that U_i(y^\sigma) > U_i(y) and U_j(y^\sigma) < U_j(y). If y_i = y_j, these two players’ expected utilities must be equal to each other’s in any equilibrium by symmetry. Hence consider i, j such that y_i > y_j without loss of generality. The case in which y_i < y_j can be shown analogously by exchanging the role of \hat{\sigma}(y) and \sigma(y). Denote \frac{2}{\delta} + 1’s expected utility in \sigma(y) and \hat{\sigma}(y) by U_q and U^\sigma_q. We know that U^\sigma_q \geq U_q \geq U_j(y) \geq U_i(y).

Consider U(y) in \sigma(y) and partition the players into three sets I_1, I_2 and I_3 = (\frac{2}{\delta} + 1) ignoring all deprived players, where I_1 is the set of players whose expected utility exceeds the expected utility of the \frac{2}{\delta}th player and I_2 is the set of players whose expected utility is the same with the \frac{2}{\delta}th player’s. Note that I_1 might be empty depending on U(y) but I_2 is never empty. I consider two cases (i) I_1 \neq \emptyset and (ii) I_1 = \emptyset.

(i) I_1 \neq \emptyset: (i-1) Suppose that i, j \in I_1 such that U_i^\sigma(y) > U_i(y) and U_j^\sigma(y) < U_j(y). Since U_i(y) = y_i + y_i \delta(\frac{1}{1 - \delta} - U_q) and U_i^\sigma(y) = y_i + y_i \delta(\frac{1}{1 - \delta} - U_q^\sigma), we have U_q > U_q^\sigma. Then for all k \in I_1, U_k^\sigma(y) > U_k(y) since U_k^\sigma(y) = y_k + y_k \delta(\frac{1}{1 - \delta} - U_q^\sigma) + \alpha_k \delta U_k(y), where \alpha_k \geq 0 is the probability that other players choose k as a coalition partner in their proposals. Thus j \notin I_1, a contradiction.

(i-2) Suppose that i \in I_1 and j \in I_2. From (i-1), we have U^\sigma_q > U_q^\sigma and U^\sigma_k(y) > U_k(y) for all k \in I_1. Let d = U_q - U_q^\sigma be the decreased expected utility of player \frac{2}{\delta} + 1 from \sigma(y) to \hat{\sigma}(y). The sum of the increased expected utility of all k \in I_1 is

\[
\sum_{k \in I_1} [U_k^\sigma(y) - U_k(y)] = \sum_{k \in I_1} [y_k + y_k \delta(\frac{1}{1 - \delta} - U_q^\sigma)] - [y_k + y_k \delta(\frac{1}{1 - \delta} - U_q)] = \sum_{k \in I_1} y_k \delta d < d.
\]

Since the sum of the increased expected utility of the members in I_1 is less than d, which is the decreased expected utility of player \frac{2}{\delta} + 1, there must be some players in I_2 whose expected utilities have increased. Note that \sum_{k \in I_1} U_k(y) = \sum_{k \in I_1} U_k^\sigma(y) = \frac{1}{1 - \delta}. Let \bar{I}_1 = \{k \in I_2 : U_k^\sigma(y) > U_k(y)\} denote the set of players who belong to I_2 and whose expected utilities in \hat{\sigma}(y) are greater than in \sigma(y). If \bar{I}_1 = I_2, we have a contradiction that j \notin I_2. Thus suppose that I_2 \setminus \bar{I}_1 \neq \emptyset. For the players k \in \bar{I}_1, the sum of the increased expected utility is

\[
\sum_{k \in \bar{I}_1} [U_k^\sigma(y) - U_k(y)] = \sum_{k \in \bar{I}_1} [y_k + y_k \delta(\frac{1}{1 - \delta} - U_q^\sigma)] - [y_k + y_k \delta(\frac{1}{1 - \delta} - U_q) + \alpha_k \delta U_k(y)] \leq \sum_{k \in \bar{I}_1} y_k \delta d,
\]

where \alpha_k is the probability that k is chosen by other players as a coalition partner in \sigma(y). Now I_1 \cup \bar{I}_1 is the set of players whose expected utilities are greater in \hat{\sigma}(y) than in \sigma(y). Compared to U(y), the sum of the increased expected utility is greater than the sum of the decreased expected utility in U^\sigma(y) by

\[
\sum_{k \in I_1 \cup \bar{I}_1} [U_k^\sigma(y) - U_k(y)] \leq \sum_{k \in I_1 \cup \bar{I}_1} y_k \delta d < d,
\]
and we have a contradiction that \( \sum_{k \in I} U_k(y) \neq \sum_{k \in I} U_k^{\sigma}(y) \).

(i-3) Suppose that \( i \in I_1 \) and \( j = \frac{n}{2} + 1 \). Then for all \( k_1 \in I_1 \), \( U_k^{\sigma}(y) > U_k(y) \) by (i-1). Also for all \( k_2 \in I_2 \), \( U_k^{\sigma}(y) \geq U_k(y) \) by (i-2). Since \( U_k^{\sigma}(y) > U_k^{\sigma}(y) \) by \( U_k^{\sigma}(y) < U_k(y) \), the probability that player \( j \) is chosen by other players as a coalition partner in \( \hat{\sigma}(y) \) is \( \hat{\alpha}_j = 1 - y_j \). Note that \( \alpha_j \), the probability similarly defined for \( j \) in \( \sigma(y) \), satisfies \( \alpha_j \leq 1 - y_j \). Thus \( \hat{\alpha}_j \geq \alpha_j \). Player \( j \)'s decreased expected utility from \( \sigma(y) \) to \( \hat{\sigma}(y) \) is

\[
U_j(y) - U_j^{\sigma}(y) = [y_j + y_j \delta(\frac{1}{1-\delta} - U_y(y)) + \alpha_j y_j U_j(y)] - [y_j + y_j \delta(\frac{1}{1-\delta} - U_y^{\sigma}(y)) + \hat{\alpha}_j y_j U_j^{\sigma}(y)]
\]

\[
= y_j \delta[U_y^{\sigma}(y) - U_{\hat{\sigma}}(y)] + \delta(y_j U_j(y) - \hat{\alpha}_j U_j^{\sigma}(y)]
\]

\[
\leq y_j \delta[U_{\hat{\sigma}}(y) - U_2(\hat{\sigma})] + \hat{\alpha}_j \delta(U_j(y) - U_j^{\sigma}(y)]
\]

\[
\leq \frac{y_j \delta}{1 - (1 - q_j) \delta}[U_{\hat{\sigma}}(y) - U_2(\hat{\sigma})],
\]

where the last inequality comes from \( -\frac{y_j \delta}{1 - (1 - q_j) \delta} < 1 \) for all \( \delta (0, 1) \). Since \( j \) is the only player whose expected utility has decreased, the total decreased expected utility between \( U(y) \) and \( U^{\sigma}(y) \) is less than the increased expected utility of player \( \frac{n}{2} \). Thus we have a contradiction that \( \sum_{k \in I} U_k(y) \neq \sum_{k \in I} U_k^{\sigma}(y) \).

(ii) \( I_1 = \emptyset \): (ii-1) Suppose that \( i, j \in I_2 \) such that \( U_i^{\sigma}(y) > U_i(y) \) and \( U_j^{\sigma}(y) < U_j(y) \). In \( \hat{\sigma}(y) \), \( i \) is not included in any other players' coalitions. Hence \( U_i^{\sigma}(y) = y_i + y_i \delta(\frac{1}{1-\delta} - U_y^{\sigma}(y)) \), whereas \( U_i(y) = y_i + y_i \delta(\frac{1}{1-\delta} - U_y(q)) + \alpha_i \delta U_i(y) \) where \( \alpha_i \geq 0 \) is \( i \)'s probability of being included in other players' coalitions in \( \sigma(y) \). Then \( U_q > U_q^{\sigma} \) follows from \( U_q^{\sigma}(y) > U_q(y) \). As in (i-2) denote the decreased expected utility of player \( \frac{n}{2} + 1 \) by \( d = U_q - U_q^{\sigma} \), and let \( \tilde{I}_1 = \{k \in I : U_k^{\sigma}(y) > U_k(y) \} \) be the set of players whose expected utility has increased in \( \hat{\sigma}(y) \) compared to the expected utility in \( \sigma(y) \). Then the total increased expected utility is

\[
\sum_{k \in \tilde{I}_1} [U_k^{\sigma}(y) - U_k(y)] = \sum_{k \in \tilde{I}_1} [y_k + y_k \delta(\frac{1}{1-\delta} - U_y^{\sigma}(y))] - [y_k + y_k \delta(\frac{1}{1-\delta} - U_y(q) + \alpha_k \delta U_k(y)])
\]

\[
\leq \sum_{k \in \tilde{I}_1} y_k \delta[U_q - U_q^{\sigma}] < d.
\]

Thus we have a contradiction that \( \sum_{k \in I} U_k(y) \neq \sum_{k \in I} U_k^{\sigma}(y) \).

(ii-2) Suppose that \( i \in I_2 \) and \( j = \frac{n}{2} + 1 \). By (ii-1), \( U_k^{\sigma}(y) \geq U_k(y) \) for all \( k \in I_2 \). Then by applying the same argument with (i-3), we have a contradiction that \( \sum_{k \in I} U_k(y) \neq \sum_{k \in I} U_k^{\sigma}(y) \).

As mentioned earlier, the case in which \( y_i < y_j \) can be shown in the same way by exchanging the role of \( \sigma(y) \) and \( \hat{\sigma}(y) \). Since (i) and (ii) cover all possible cases, we concede that there exists no player whose expected utility from \( y \) is different in any distinct equilibria \( \sigma \) and \( \hat{\sigma} \).

**Proof of Proposition 2.** For each state in \( \Delta_t \), let \( \mu(s) \) be such that any active player \( i \) proposes to take the entire dollar. Without fully specifying the acceptance sets, it is enough to state that all players acceptance sets \( \mathcal{A}_i(s) \) includes the proposals that are weakly better than the status quo and all deprived players acceptance sets equal to \( X \).

For each state in \( \Delta_m \), construct \( \mu(s) \) by setting \( \mu_i(y^{\phi}|s) = \pi_i^{\phi}(U^{\phi}) \), where \( \pi_i^{\phi} \) is \( i \)'s SSPE proposal.
strategy of the state game $\Gamma(s)$ and $y^i$ is an allocation such that $U(y^i) = U^i$. For any $s \in \Delta_m$ and $U^i$ such that $\pi^s_i(U^i) > 0$, we know that $U^i$ has at most $\frac{s}{2} + 1$ positive entries since each proposer $i$ selects only a minimum number of costly coalition partners. Then by Lemma 6, there exists $y^i \in X$ such that $U(y^i) = U^i$ even though it may not be unique. For the purpose of this proof, it is enough to assume that each proposer $i$ choose a single $y^i \in \{x \in X : U(x) = U^i\}$ for each $U^i$ such that $\pi^s_i(U^i) > 0$, but allowing each proposer to choose finite but multiple proposals has no effect. By Lemma 7, further, the set $Y^i = \{y \in X : U(y) = U^i\}$ remain the same in all equilibria. Thus, if $\mu$ and $\mu$ are two distinct equilibrium proposal strategies, $\mu(s)$ and $\mu(s)$ does not affect $U(y^i)$ for any given $y^i$, and the expected utility vector from $y^i$, $U(y^i)$, is the same in $\mu(y^i)$ and $\mu(y^i)$.

Given these proposal strategies $\mu(i|s) = \pi^{s}$ for all $i$ and $s \in \Delta_m$, each player’s expected utility from the status quo is given by

$$U_i(s) = s_i + \delta \left[ s_i \sum_{y^i : \mu_i(s) > 0} \mu_i(y^i|s)U_i(y^i) + \sum_{j \neq i} s_j \sum_{y^i : \mu_j(s) > 0} \mu_j(y^i|s)U_j(y^i) \right] = \psi^{s},$$

where $\psi^{s}$ is the SSPE reservation value vector in the state game $\Gamma(s)$. Now, every player $j$’s SSPE voting strategy of accepting any $U$ if and only if $U_j \geq \psi^j$ in the state game $\Gamma(s)$ is equivalent to accepting any $y$ if and only if $U_j(y) \geq U_j(s)$. Every proposal $y^i$ of proposer $i$ is approved by a set of costly coalition partners $L^i_0$ which constitutes a decisive coalition together with all deprived players and $i$ since $U_j(y^i) = U_j^i = \psi^i = U_j(s)$ for all $j \in L^i_0 \in L^i$. Also, $\mu_i(s)$ is optimal for all $i$ since every $y^i$ in the support of $\mu_i(s)$ is associated with a set of costly coalition partners $L^i_0$ that solves the problem $\min_{L \in \bar{L}_i} \sum_{j \in L} \psi^j$, which is equivalent to $\min_{L \in \bar{L}_i} \sum_{j \in L} U_j(s)$. It is obvious from the above discussion that thus constructed $\sigma = \times_{i \in I} (\sigma_i(s))_{s \in S}$ satisfies all conditions in Proposition 1 and $\sigma = (\mu, A)$ is an equilibrium. ■

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6Precisely, we allow $i$ to use a set of finite but multiple proposals $\{y^i|s\}$ such that $U(y^i|s) = U^i$ for all $\eta = 1, \ldots, ||\eta||$. Then the rest of the proof is identical when setting $\sum_{\eta \in \eta} \mu_i(y^i|s) = \pi^s_i(U^i)$. 

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