The Emergence and Persistence of Oligarchy: 
A Dynamic Model of Endogenous Political Power

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Abstract

In this article, I present a formal model in which multiple groups within a society bargain over an object that influences their future bargaining power. Interpreting the bargaining object as wealth, this model provides a theoretical framework to examine the long-run consequences of the interdependency between economic and political power. Alternatively, by interpreting the bargaining object as political institutions that affect future power of relevant political actors, the model provides a theory of endogenous evolution of political institutions. The model of endogenous political power predicts extreme concentration of resources and power—permanent tyranny of a single player or a permanent oligarchy of two players monopolizing the entire resources and power. These tyrants or oligarchs perpetuate their domination by deterring all other players from the subsequent political process. The long-run outcome depends on the distribution of resources and power in the initial state given that all players are perfectly farsighted. I also discuss the importance of farsightedness in preventing the onset of tyranny.

keywords: dynamic bargaining, Markov perfect equilibrium, endogenous political power, oligarchy
1 Introduction

In this article, I present a formal model in which multiple groups within a society bargain over an object that influences their future bargaining power. Interpreting the object as economic resources, this model captures the situations in which those who have exert their economic power to obtain greater political power, and then the resource distribution is reconsidered under the stronger influences of the wealthy. From Montana’s William A. Clark who bought out a seat in the U.S. Senate in the early twentieth century to ever increasing campaign donations by big firms today, we have numerous examples showing the influence of economic elites in the political process. Incorporating the idea “wealth begets power, which begets wealth,” (Stiglitz, 2011) this article thus provides a theoretical framework to examine long-run consequences of the increasing inequality we witness today.

This model applies to many other situations in which the bargaining outcome affects the conditions on which relevant actors resume the bargaining process again. As an example, consider territorial conflicts among nation states or ethnic groups. Territorial conflicts do not exclusively concern the territory itself. The actors in dispute claim their jurisdiction over natural resources, the residents of the territory as potential labor and military forces and any strategic value the location may have—all of which enhance the winner’s military power (Fearon, 1996). In the absence of a supranational entity, the outcomes of such interstate bargaining over how to divide territory are not stable in nature, and the territory is subject to recurring negotiations under the threat of war. Simply put, the bargaining outcome on territorial division today influences the relative military power of the nations tomorrow, and the bargaining process repeats until they reach a stable resolution.

Most broadly, this model can be interpreted as a theory of the endogenous development of political institutions and stable ruling coalitions. In previous literature on political institutions, the emphasis has been put on identifying the consequences of institutions. Taking institutions as exogenous variables, these studies investigate how institutions shape strategic interactions between political and economic actors, party systems and government policies. Yet institutions are usually selected by the very actors who are supposed to be constrained by them in the future (Diermeier and Krehbiel, 2003; Greif and Laitin, 2004; Riker, 1980; Shepsle, 1989). In other words, institutions that determine subsequent allocations of political power are the object the same actors negotiate on. Anticipating the consequences associated with different arrangements of institutions, political actors seek to manipulate the rules during their creation in order to gain an advantage in the future. A considerable array of studies characterize institutional reforms as continual bargaining processes and provide abundant evidence from both new and advanced democracies that political actors are indeed aware that their future political power is contingent on the current bargaining outcomes (Bawn, 1993; Benoit and Hayden, 2004; Boix, 1999; Brady and Mo, 1992; Diaz-Cayeros and Magaloni, 2001; Ishiyama, 1997; Kamiński, 1999; Luong and Weinthal, 2004; McElwain, 2008; Remington and Smith, 1996). As an example, the Polish Roundtable Agreement in 1989 specified the structure of the new parliament, legislative procedures and the electoral rules to be applied to the first elections. This institutional arrangement mirrored the expected power configuration in the new regime, which was acceptable to and preferred by the main participants of the negotiation—the ruling communist party (the Polish United Workers’ Party) and the opposition representative (Solidarity). They anticipated that the PUWP would maintain its dominance in executive and legislative matters and Solidarity would securely take a legalized opposition position after the first elections (Kamiński, 1999). The scope of institutional manipulation is not limited to large-scale rules such as electoral systems and the constitutional allocation of powers. McElwain (2008) convincingly shows that the Japanese Liberal Democratic Party prolonged its electoral dominance by manipulating micro-level electoral rules: the length of the election campaign periods and the management of campaign finance.\(^1\)

\(^1\)The rules concerning qualifications of voters and candidates have also been frequently used as a means for controlling subsequent power structure (Manin, 1997).
interpreting the feasible resource allocations in the model as feasible institutional arrangements, this article provides an analytical framework to study the endogenous evolution of institutions in societies with weak or unstable institutions.

Theoretically, my model contributes to the literature on dynamic bargaining with endogenous status quo. In the traditional bargaining framework, originated from Baron and Ferejohn (1989) and Rubinstein (1982), the bargaining process ends once the players reach an agreement. In some policy area, however, the actors have opportunities to reconsider existing policies, and any chosen policy has effects until it is replaced by another policy. A growing number of studies recently have developed theoretical models applicable to these continuing programs (Anesi, 2010; Baron, 1996; Baron and Bowen, 2013; Diermeier and Fong, 2011; Duggan and Kalandrakis, 2012; Kalandrakis, 2004, 2009; Nunnari, 2011; Penn, 2009). My model builds on Kalandrakis (2004, 2009). Kalandrakis analyzes a multimember distributive bargaining game with endogenous status quo under a fixed proposal and voting rule and shows that the proposer in each period extracts all resources in the long run. Nunnari (2011), considering a similar game but with a veto player, shows that when there is an advantaged player in terms of voting power, that player eventually extracts almost all resources. Diermeier and Fong (2011) in turn examines the case with an advantaged proposer. Assuming the presence of a persistent agenda setter, they show that the agenda setter’s advantage may be limited despite the greater institutional prerogative she enjoys. Non-agenda setters protect other players’ interests to prevent further exploitation by the persistent agenda setter. Although these studies have greatly enhanced our understanding of dynamic bargaining processes in a more realistic context, most of them commonly assume that the proposal and voting power of the players are exogenously given and fixed regardless of bargaining outcomes.

My model differs from the aforementioned studies in that the proposal and voting rules evolve over time depending on the resource distribution plan the players agreed on in the previous period. More specifically, I study an infinite horizon multimember divide-the-dollar game with endogenous status quo and endogenous recognition and voting rules. I focus on perfectly farsighted players. In each period a proposer is selected according to a recognition rule and the proposal is passed if it obtains more than half of total voting weights. If the new proposal is passed, the players receive one’s share of the dollar prescribed in the proposal. When the players fail to reach an agreement in a given period, the status quo carried over from the last period is automatically implemented. Before moving on to the next period, the players’ recognition probability and voting weights are adjusted according to the share of the dollar each player received in the last period. To preview the result, the long-run outcome of the game is either a tyranny of a single player who monopolizes the wealth and political power or an oligarchy of two players who share total wealth and power equally. In either case, the tyrant or oligarchs deter the entry of other players and perpetuate their rules in all subsequent periods. The long-run outcome depends on the initial distribution of wealth and power. If there is an overwhelmingly strong player at the beginning of the game, that player becomes a tyrant. Otherwise, players never agree to other players’ dictatorship in the expectation that they might be an oligarch in the future and receive half of the dollar forever.

The rest of the paper is organized as follows: In Section 2, I introduce the bargaining environment, necessary notation and the equilibrium concept. In Section 3, I characterize the dynamics in equilibria along with the existence of equilibrium. In Section 4, I discuss an alternative model in which players are not perfectly farsighted and the policy space is discrete. I show that the main results are preserved when the players sufficiently farsighted. Finally I summarize the results and conclude. All proofs are relegated to online Supporting Information.
2 Model

Setup A set of players, $I = \{1, \ldots, n\}$ with $n \geq 3$, divide a dollar for time $t = 1, \ldots, \infty$. The bargaining outcome at time $t$ is denoted by $x'$. Let $X = \{x \in \mathbb{R}^n| \sum_{i \in I} x_i = 1\}$ and $x_i \geq 0$ be the set of all feasible divisions of the dollar. In each period $t \geq 1$, the bargaining environment is summarized by $E' = (x'^{-1}, p', w')$, where the status quo $x'^{-1}$ is the bargaining outcome carried over from period $t-1$, and $p'$ and $w'$ are length-$n$ vectors specifying each player’s recognition probability and voting weights, respectively. The parameters representing the players’ bargaining power, $p'$ and $w'$, change over time endogenously to the previous bargaining outcome: $p' = w' = x'^{-1} \in X$. That is, the proposal and voting power of the players are proportional to their resources received in the previous period. Since $p'$ and $w'$ are redundant, I denote the environment with status quo $x'^{-1}$ as $E'(x'^{-1})$. The status quo in the initial state $x^0$ is exogenously given, and $E^1(x^0)$ is realized according to $x'^0$.

The timing of the game is as follows: The bargaining at period $t$ begins in the environment $E^t(x'^{-1})$. At the beginning of period $t$, one of the players is selected as a proposer according to the recognition rule $p'$ and makes a proposal $y \in X$. Observing the proposal $y$, all players simultaneously vote whether to accept or reject it by a weighted voting rule. If the sum of the voting weights in favor of the proposal is greater than one half, the proposal $y$ is adopted as the $t$-period policy $x'$; otherwise the status quo $x'^{-1}$ becomes $x'$. At end of the period, each player receives one’s share of the dollar as prescribed by $x'$. The bargaining at time $t+1$ begins in the new environment $E^{t+1}(x')$, and the same process continues ad infinitum.

Strategies Throughout the analysis I focus on stationary Markov strategies. In stationary Markov strategies, the players ignore the complicated history of the past plays, as far as the payoff relevant parameters are identically given at the end of different histories (Maskin and Tirole, 2001). In the bargaining environment described above, players’ actions remain the same if the status quo division and the distribution of proposal and voting power among the players are the same. That is, the players use the same actions at time $t$ and $t'$ if $E' = E''$, which is equivalent to $x'^{-1} = x''^{-1}$. Dropping the time index and simplifying the notation, define the state $s$ with status quo $x'^{-1} = x''^{-1} = x$ by setting $s = x$. Let $S$ be the set of all states, then we have $S = X$.

Player $i$’s strategy is a pair of proposal strategy $\mu_i$ and acceptance set $A_i$. Player $i$’s proposal strategy is a map $\mu_i : S \rightarrow \Delta X$, where $\Delta X$ is the set of probability distributions over $X$. For any state $s$ and feasible division $y \in X$, $\mu_i(y|s)$ dictates the probability that player $i$ proposes $y$ in state $s$. I assume that the support of $\mu_i$ is finite for all $i \in I$ and for all $s \in S$, avoiding measurability issue. In case $\mu_i(y|s) = 1$, I denote $\mu_i(s) = y$ with slight abuse of notation. The acceptance set $A_i : S \Rightarrow X$ represents $i$’s voting strategy, where $A_i(s)$ is the set of proposals $i$ will accept in state $s$. Let $\sigma_i \equiv (\mu_i, A_i)$ denote player $i$’s Markov strategy and let $\sigma = \times_{i \in I} \sigma_i \in \times_i \Sigma_i = \Sigma$ be a generic Markov strategy profile, where $\Sigma_i$ is the set of all Markov strategies of player $i$.

Since players are ex ante identical in ways other than the status quo share, recognition probability and voting weights, symmetry is a natural assumption for strategies. Define a one-to-one and onto function $\phi : I \rightarrow I$, and for any $x \in X$, let $\hat{\phi} : X \rightarrow X$ be a map such that $\hat{\phi}(x) = (x_{\phi(1)}, \ldots, x_{\phi(n)})$. A Markov strategy profile $\sigma$ is symmetric if for any $x$ and its $\hat{\phi}$-permutation $\hat{\phi}(x)$, $\sigma_i(\hat{\phi}(x)) = \sigma_{\hat{\phi}(i)}(x)$ for every $i$. In symmetric Markov strategies, players are not identified by their names. Two players having identical status quo shares in a given state plan to propose and vote symmetrically, and all other players treat these two players identically.

Preferences Each player’s short-term utility from $t$-period policy $x'$ equals one’s share of the dollar: $u_i(x') = x'_i$. For a sequence of bargaining outcomes $\{x'^t\}_{t=1}^T$, player $i$’s $T$-period utility is the average sum of
All players are perfectly farsighted and do not differentiate current and future payoffs as long as their sum remains the same.

A Markov strategy profile \( \sigma \) prescribes how all players behave in every state and thus assigns a probability distribution over the set of all possible sequences of outcomes for \( t = 1, \ldots, \infty \) from any state \( s \). Let \( \mathcal{L}_s \) be the set of all decisive coalitions in state \( s \): \( \mathcal{L}_s = \{ L \in 2^I | \sum_{i \in L} s_i > 1/2 \} \). Define the social acceptance set in state \( s \) by \( A(s) = \bigcup_{L \in \mathcal{L}_s} \bigcap_{i \in L} A_i(s) \). The social acceptance set contains all divisions that will be approved by at least one decisive coalition in state \( s \).

Let \( v^\sigma_{i, \tau - 1}(x) \) be player \( i \)'s \( \tau \)-period continuation value given that the bargaining outcome in the first period is \( x \) and all players henceforth play the game up to period \( \tau \) according to \( \sigma \):

\[
v^\sigma_{i, \tau - 1}(x) = \sum_{j \in I} x_j \sum_{y: \mu_j(y|x) > 0} \mu_j(y|x) \left[ \frac{u_i(y) + (T - 2)u^\sigma_{i, \tau - 2}(y)}{T - 1} \right] I_{A(x)}(y) \]

\[
+ \sum_{y: \mu_j(y|x) > 0} \mu_j(y|x) \left[ \frac{u_i(x) + (T - 2)u^\sigma_{i, \tau - 2}(x)}{T - 1} \right] I_{A \setminus A(x)}(y),
\]

where for any set \( A \subseteq X \) and \( x \in X \), \( I_A(x) = 1 \) if \( x \in A \) and \( 0 \) otherwise. Intuitively, \( i \)'s \( \tau \)-period continuation value is the average expected utility from period 2 to \( \tau \), excluding period 1, before the identity of the period 2 proposer is known. In the first part of equation (1), each player \( j \in I \) is recognized with probability \( x_j \) and proposes \( y \in X \) with probability \( \mu_j(y|x) \). If \( y \) is in the social acceptance set, \( i \) receives instantaneous utility \( u_i(y) \) and expects to receive the continuation value from the new state \( y \) for the remaining \( T - 2 \) periods. The sum of \( u_i(y) \) and \( (T - 2)u^\sigma_{i, \tau - 2}(y) \) is averaged over \( T - 1 \) periods. In the second part of the equation, player \( j \) proposes a socially unacceptable division \( y \) with probability \( \mu_j(y|x) \). Then the status quo \( x \) is implemented and gives player \( i \) instantaneous utility \( u_i(x) \) and the continuation value \( v^\sigma_{i, \tau - 2}(x) \) for the remaining \( T - 2 \) periods, which is averaged over \( T - 1 \) periods.

Using \( v^\sigma_{i, \tau - 1}(x) \), we define \( i \)'s \( \tau \)-period expected utility from a successful proposal \( x \) in \( \sigma \):

\[
U^\sigma_{i, \tau}(x) = \frac{u_i(x) + (T - 1)v^\sigma_{i, \tau - 1}(x)}{T}.
\]

Player \( i \) receives \( u_i(x) \) in the current period and expects to receive her continuation value from state \( x \) for the \( T - 1 \) remaining periods beginning from period 2, which is averaged over \( T \) periods.

Since \( T \to \infty \), I define players’ long-term preferences by applying the overtaking criterion (Rubinstein, 1979). For \( x, y \in X \), \( i \) prefers \( x \) to \( y \) (\( x \succeq_i y \)) in \( \sigma \) if and only if,

\[
\lim_{T \to \infty} T[ U^\sigma_{i, T}(x) - U^\sigma_{i, T}(y) ] \geq 0.
\]

Given any Markov strategy profile \( \sigma \), a player prefers \( x \) to \( y \) if \( x \) gives a higher expected utility than \( y \) when \( T \to \infty \).

Under the overtaking criterion, player \( i \) strictly prefers \( x \) to \( y \) whenever \( \lim_{T \to \infty} U^\sigma_{i, T}(x) > \lim_{T \to \infty} U^\sigma_{i, T}(y) \). If \( \lim_{T \to \infty} U^\sigma_{i, T}(x) = \lim_{T \to \infty} U^\sigma_{i, T}(y) \), the overtaking criterion takes account of the payoff differences in the finite number of initial periods. To get an intuition, consider the following streams of

\[\footnote{In principle, the usual limit may not be well-defined for some \( x, y \in X \) and in \( \sigma \in \Sigma \), which requires the use of the limit inferior or a Banach limit. As will be clear below, however, the usual limit exists for any pair \( x, y \in X \) in any equilibrium strategy.} \]
payoffs of $i$ from $x, y, z$:

$x : (\frac{1}{3}, \frac{1}{3}, 0, 0, \ldots) \quad y : (\frac{1}{3}, 0, 0, 0, \ldots) \quad z : (0, 0, 1, \frac{1}{3}, \frac{1}{3}, \ldots)$.

Since $U_i(x) = \frac{2}{3T}, U_i(y) = \frac{1}{3T}$, and $U_i(z) = \frac{T-2}{3T}$ for any $T \geq 2$, we have $z \succ_i x \succ_i y$ under the overtaking criterion. Comparing $x$ and $z$, the average payoff over an infinite number of periods from $x$ is zero but the average payoff from $z$ is $\frac{1}{3}$. Between $x$ and $y$, these two streams of payoffs both give zero average payoff when $T \to \infty$, but $\lim_{T \to \infty} T[U_i(x) - U_i(y)] = \lim_{T \to \infty} T \cdot \frac{1}{3T} = \frac{1}{3} > 0$.

**Equilibrium notion** I seek for Markov perfect equilibria with several refinement conditions.  

**Definition 1.** A stationary Markov strategy profile $\sigma^* = \{(\mu^*_i, A^*_i)\}_{i=1}^n$ is a symmetric stationary Markov perfect equilibrium in stage-undominated voting strategies if, for all $i \in I$, $s \in S$, and $\hat{\phi} : X \to X$,

\[
\begin{align*}
\sigma_i(\hat{\phi}(s)) &= \sigma_{\phi(i)}(s) \\
y \in A^*_i(s) &\iff y \succeq_i s \\
\mu^*_i(y) > 0 &\iff y \in \{x \in A^*(s) : x \succeq_i x', \forall x' \in A^*(s)\}. \\
s \sim_i y, \forall y \in A(s) &\Rightarrow \mu_i(s) = s.
\end{align*}
\]

Except for (6), the definition is from Kalanderakis (2004). Condition (3) states that the players use symmetric strategies. Condition (4) requires that the players do not vote for the proposals that would give lower expected utility than the status quo in equilibrium. This stage-undominated voting requirement eliminates a number of uninteresting equilibria. Without (4), we can easily construct an equilibrium in which every player votes for any proposal and the proposer in the first period immediately takes the whole dollar from most states. Condition (5) requires that players make optimal proposals in the social acceptance set. Condition (6) says that any proposer indifferent between all alternatives in the social acceptance set proposes the status quo division instead of randomly selecting other divisions. As a tie-breaking rule in proposal strategies, it can be interpreted as assuming a small cost of forming a new coalition on behalf of the coalition formateur. If a coalition formateur (a proposer) expects no extra benefit from altering the status quo, he has no reason to exert the efforts necessary to form a new coalition. As long as there is at least a single division in the social acceptance set that is strictly better than the status quo to the proposer, the proposer chooses that division over the status quo. This tie-breaking rule does not affect other players’ voting behavior. Henceforth, I refer to the equilibrium defined above simply as an *equilibrium*.

**Partitions of states** To present the results more concisely, I partition the state space into several sub-spaces according to the level of concentration of voting power.

**Definition 2.** State $s$ is in the set of

(i) tyrannical states $S_T$ if $\exists i \in I$ such that $s_i = 1$;

(ii) dictatorial states $S_D$ if $\exists i \in I$ such that $[i] \in \mathcal{L}_s$ and $s \notin S_T$;

(iii) oligarchic states $S_{OL}$ if $\exists L \in \mathcal{L}_s$ such that $\forall i \in L, i$ has a veto and $s \notin (S_T \cup S_D)$;

\[3\text{The limit of the means criterion is also frequently employed in infinite horizon games without discounting. In the limit of the means criterion, } x \sim_{LM} y \iff \lim_{T \to \infty}[U_i(x) - U_i(y)]. \text{ In the above example, } x \sim_{LM} y \text{ because } \lim_{T \to \infty} \frac{2}{3T} = 0. \text{ The overtaking criterion is more discriminative among alternatives. With respect to the equilibrium analysis, a player becomes indifferent between all allocations in which his share is more than a half under the limit of the means, whereas the overtaking criterion differentiates most of those alternatives.}

\[4\text{The assumptions that the players make only the proposals in the social acceptance set is introduced to simplify the analysis, but dropping the assumption has no effect on the results.} \]
(iv) collegial states $S_{COL}$ if $\bigcap_{L \in \mathcal{L}} L \neq \emptyset$ and $s \notin (S_T \cup S_D \cup S_{OL})$; and (v) noncollegial states $S_{NC}$ if $\bigcap_{L \in \mathcal{L}} L = \emptyset$.

In Definition 2 (iii), $i$ has a veto if $x \notin A_i(s)$ implies $x \notin A(s)$. In words, if player $i$ has a veto, he can prevent the passing of any proposals that he dislikes. Figure 1 illustrates the partitions of states in a three-player game. The points in the triangle correspond to particular status quo divisions, where the bottom left corner is $(1, 0, 0)$, the bottom right corner is $(0, 1, 0)$ and the center-top corner is $(0, 0, 1)$. The three states with status quo divisions $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are tyrannical as seen in Figure 1 (a). In dictatorial states (b), one of the players has more than half of the dollar, as seen in the shaded areas and solid lines. Oligarchic states in (c) consist of the three points in which two of the players divide the dollar equally. In the definition above, two players share the dollar equally in any oligarchic state regardless of the number of players. Figure 1 (d) illustrates collegial states. In collegial states, there is a single player whose status quo share is one half, and there are at least two players possessing positive status quo shares regardless of the number of players. In non-collegial states (e), every player’s share is less than a half. Thus, there is no player who is in every decisive coalition.

Since $S = X$, I partition $X$ into $X_T, X_D, X_{OL}, X_{COL}$ and $X_{NC}$ according to the state they induce in the subsequent period. If $s = x \in S_D$, then $x \in X_D$, etc. In the analysis below, I denote the set of dictatorial divisions in which player $i$ is the dictator by $X_D$ and the set of collegial divisions in which $i$’s share is one half by $X_{COL}$. Similarly, $X_{D,i}$ is the set of states in which player $i$ is not a dictator.

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The term “tyranny” is borrowed from Jordan (2006). The definitions of dictatorial, oligarchic, collegial and noncollegial states are consistent with the definitions of voting rules in the social choice literature except for the slight modification introduced to make them mutually exclusive (Austen-Smith and Banks, 2000).
3 Analysis

In this model of endogenous power, the main challenge is to pin down players’ preferences over different allocations. For instance, consider any state $s$ and player $i$. Player $i$’s $T$-period expected utility depends on equilibrium proposals of all players in $s$. In order to find the equilibrium proposals of any player in $s$, we should know the acceptance sets of all other players in $s$ and be able to compare the expected utilities of the proposer from all allocations in the social acceptance set, which generates infinite chains of allocations to evaluate. My analysis of the game begins from the states with the most concentrated power structure and proceeds in descending order. As will be seen, this approach significantly simplifies the aforementioned complexity of the players’ long term utility assessments on different divisions.

Lemma 1 through Lemma 4 present players’ equilibrium proposal and voting behavior in tyrannical, dictatorial, oligarchic, collegial and non-collegial states, assuming that equilibrium exists. Proposition 1 then establishes equilibrium existence based on these lemmas and characterizes the long-run outcomes and dynamics in equilibrium including the advantage of proposers in the initial periods.

Lemma 1. In tyrannical, dictatorial and oligarchic states, equilibrium proposals are uniquely given as follows:

(i) In tyrannical states, the tyrant proposes the status quo division.

(ii) In dictatorial states, the dictator proposes his tyrannical division and other players offer the status quo share to the dictator and take the remainder.

(iii) In oligarchic states, the oligarchs propose the status quo division.

All proposals in (i), (ii) and (iii) are accepted by a decisive coalition.

A tyrant, whose status quo share is the entire dollar, never shares his wealth and power with other players and continues to hold the entire wealth and power. Thus the $T$-period expected utility of tyrant $i$ from his tyrannical division $s$ is given by $U^T_i(s) = \lim_{T \to \infty} U^T_i(s) = 1$. A tyrant never proposes another allocation. The instantaneous utility $u_i(y)$ is strictly less than 1 and the continuation value cannot exceed 1, which implies $U^T_i(y) < 1$ for any $y \neq s$. Thus, $\lim_{T \to \infty} T[U^T_i(s) - U^T_i(y)] > 0$ for all $y \neq s \in X$.

In dictatorial states, the dictator, whose status quo share is greater than a half, constitutes a decisive coalition by oneself. Hence he proposes to take the entire dollar by himself when recognized as a proposer and passes his tyrannical division plan alone. Other players vote against that proposal but do not possess enough voting power to stop the transition to tyranny. If recognized to propose, non-dictators in dictatorial states make proposals that guarantee the status quo share to the dictator and assign the remainder since the dictator does not accept any smaller shares than the status quo. If the initial state is dictatorial, therefore, there remain at most two players, the dictator and the proposer, having positive shares of the dollar after a single round of bargaining, regardless of the number of players having positive shares at the beginning. Recognition of the dictator as a proposer induces tyranny in the next period. For the dictator, protecting his status quo share is important not only because it affects today’s wealth but also, more importantly, because it determines how likely he is recognized as a proposer in the next period and how fast he can turn into a tyrant.

For oligarchic states, notice that a state is oligarchic if and only if there is a decisive coalition in which every member has a veto. Since the passage of a proposal requires more than half of the total voting weights in favor of the proposal, a player has a veto if and only if one’s voting weight is one half or more. Also, the definition of oligarchic states excludes the states in which there is a decisive coalition of a single player (dictatorial or tyrannical states). Thus there are two players who equally share the entire dollar in any oligarchic states. These two players in the oligarchy monopolize the proposal power, and each of them has a veto. Neither of the oligarchs can pass one’s dictatorial division because the other oligarchy member, who has a veto, opposes the plan. These oligarchs do not distribute any wealth and power to a third member outside the oligarchy, either. To pass a new division, the proposer has to guarantee the current expected
utility to the other oligarchy member. Accordingly, allocating a positive amount to a third member means a decrease in the proposer’s own expected utility if such a plan were to pass. Once an oligarchy is in place, all other players are left without wealth and power forever, and the oligarchy remains for all the subsequent periods in equilibrium. In an oligarchic state \( s \), each oligarch \( i \)'s \( T \)-period expected utility is

\[
U_i^T(s) = \lim_{T \to \infty} U_i^T(s) = \frac{1}{2}. \quad (7)
\]

Before proceeding, it is convenient to establish a non-dictator \( i \)'s expected utility from another player’s dictatorship. Suppose that \( s \in X_{Dd} \) with \( s_i \in (0, \frac{1}{2}) \) and \( s_d = 1 - s_i \). In state \( s \), player \( d \) is the dictator and the sum of \( i \) and \( d \)'s status quo shares is 1. Given the equilibrium proposals in Lemma 1 (ii), non-dictator \( i \)'s \( T \)-period expected utility from \( s \) is

\[
U_i^T(s) = \frac{s_i + \cdots + s_i^T}{T} = s_i(1 - s_i^T) / T(1 - s_i). \quad (8)
\]

Non-dictator \( i \) receives instantaneous utility \( s_i \) in the current period and receives \( s_i \) if \( i \) is recognized as a proposer with probability \( s_i \) in period 2. If \( i \) is the proposer in the second period, \( i \) again expects to receive \( s_i \) when he is recognized as a proposer in period 3 with probability \( s_i \) and so on.

By \( \sum_{k \in I} U_k^T(s) = 1 \), dictator \( d \)'s \( T \)-period expected utility from \( s \) is

\[
U_d^T(s) = 1 - s_i(1 - s_i^T) / T(1 - s_i). \quad (9)
\]

From (8) and (9), we see that for any \( s \in X_{Dd} \) with \( s_i \in (0, \frac{1}{2}) \),

\[
\lim_{T \to \infty} U_i^T(s) = 0 \quad \text{and} \quad \lim_{T \to \infty} U_d^T(s) = 1. \quad (10)
\]

The dictator’s expected utility in the limit converges to 1 even if \( s_d \) is only slightly larger than \( \frac{1}{2} \). The dictator turns into a tyrant as soon as he becomes a proposer and henceforth receives 1 for all the subsequent periods. Similarly, a non-dictator’s expected utility is zero in the limit even if \( i \)'s current share is only slightly smaller than \( \frac{1}{2} \). Also notice that \( U_i^T(s) \) in (8) is increasing in \( s_i \) and \( U_d^T(s) \) in (9) is decreasing in \( s_i = 1 - s_d \). Thus, for \( x, y \in X_{Dd} \) with \( 0 < y_i \leq x_i < \frac{1}{2} \) and \( x_d = 1 - x_i, y_d = 1 - y_i \), non-dictator \( i \) prefers \( x \) to \( y \) by

\[
\lim_{T \to \infty} T[U_i^T(x) - U_i^T(y)] = \lim_{T \to \infty} T\left[\frac{x_i(1 - x_i^T)}{T(1 - x_i)} - \frac{y_i(1 - y_i^T)}{T(1 - y_i)}\right] = \frac{x_i}{1 - x_i} - \frac{y_i}{1 - y_i} \geq 0,
\]

and dictator \( d \) prefers \( y \) to \( x \) similarly.

The next lemma characterizes the players’ equilibrium behavior in collegial states. In collegial states, there is a player whose status quo share is exactly \( \frac{1}{2} \) and there are at least two other players with positive status quo shares. I call the player whose share is \( \frac{1}{2} \) as a collegium player and all other players having positive status quo shares as noncollegium players, without mentioning the players having a zero share.

**Lemma 2.** The equilibrium proposals in collegial states are uniquely given as follows: the collegium player proposes the status quo and all noncollegium players propose an oligarchic division from which the collegium player and the proposer form an oligarchy. All proposals are accepted by a decisive coalition.

According to the proposal and voting plans in collegial states stated in Lemma 2, collegium player \( c \)’s
expected utility from \( s \) is \( U^T_i(s) = \frac{1}{2} \). For noncollegium player \( i \),

\[
U^T_i(s) = \frac{1}{T}[u_i(s) + (T - 1)v^{-1}_i(s)] = \frac{1}{T}[s_i + \frac{(T - 1)(U^T_i(s) + s_i)}{2}] = s_i. \tag{11}
\]

Combining (10) and (11), we see that \( \lim_{T \to \infty} T[U^T_i(s) - U^T_i(y)] > 0 \) for all \( s \in X_{COL_i} \) and \( y \in X_{D_i} \). In words, noncollegium players strictly prefer status quo \( s \) to all other player’s dictatorial divisions, and none of the noncollegium players accept other players’ dictatorial proposals. In equilibrium, noncollegium players envision the possibility, however small it is, to be an oligarch and receive \( \frac{1}{2} \) for an infinite number of remaining periods. Accordingly, they are not willing to exchange this long-term possible gains for a short-term gain and reject all dictatorial proposals if proposed. Being aware of the preferences of noncollegium players against dictatorial divisions, the collegium player becomes indifferent between all socially acceptable divisions and thus proposes to maintain the current wealth and power distribution instead of forming an alternative coalition due to condition (6). In sum, perfectly farsighted players never invite a dictator or tyrant in equilibrium, and the long-run outcome is always a permanent oligarchy of two players if the initial state is collegial.

The next lemma characterizes farsighted players’ voting behavior in noncollegial states.

**Lemma 3.** There is no transition from noncollegial states to dictatorial or tyrannical states in equilibrium.

If the players have relatively equal wealth and power in the initial state, the endogenous power bargaining process never reaches a dictatorship. As in collegial states, none of the players in noncollegial states accept other players’ dictatorial proposals, and it imposes an endogenous limit to the extent to which proposers exploit other players in noncollegial states. The key intuition to this result is that in all noncollegial states every player having a chance to be a proposer can exchange one’s bargaining position with any other player’s position through one’s own proposal. To see this, consider noncollegial state \( s \) and two players, \( i, j \in I \), both having positive status quo shares. By the definition of the acceptance set, \( s \in A_k(s) \) for every \( k \in I \). Now consider \( x \in X_{NC} \) such that \( x_j = s_i \), \( x_i = s_j \), and \( x_k = s_k \) for all \( k \neq i, j \). By symmetry, \( x \in A_k(s) \) for all \( k \neq i, j \), and \( x \in A_i(s) \) or \( x \in A_j(s) \) or both. Thus player \( j \) can propose \( x \), not necessarily optimally but at least successfully, and exchange the bargaining position with \( i \) with the support of all players except \( i \). Symmetry, again, implies that \( j \)'s expected utility from \( x \) is the same with \( i \)'s expected utility from \( s \). Thus if there is a player who can possibly become a dictator in the future, all other players also envision such a possibility to become a dictator at least within two rounds in expectation. Accordingly, the players prefer receiving a smaller short-term utility and staying in a non-dictatorial state in which the expected utility from the status quo is strictly positive in the limit to receiving a relatively larger short-term utility but enabling transition to another player’s dictatorship in which the long-term expected utility is zero in the limit.

From Lemmas 2 and 3, we know that \( U^T_i(s) = \frac{1}{2} \) if and only if \( s_i = \frac{1}{2} \) for any \( i \). First, every player’s \( T \)-period expected utility is strictly less than \( \frac{1}{2} \) in any noncollegial state. In noncollegial state, every player’s status quo share is strictly less than \( \frac{1}{2} \) and no one can expect to receive more than \( \frac{1}{2} \) at any time in the future because there is no transition to dictatorship from noncollegial or collegial states. In dictatorial states, the dictator receives more than a half and the non dictators receive less than a half in all subsequent periods, which makes none of their expected utilities equal to one half. Noncollegium players’ expected utilities are also strictly less than one half as given in (11).

A natural question arises concerning the stable wealth and power distributions in the long run and the dynamics on the equilibrium paths to reach those stable states if those exist. In noncollegial states, there is no single player whose wealth and power overwhelms the wealth and power of other players. In other words, the players in a noncollegial state are relatively equal to each other both in terms of their resources under possession and political power to initiate future changes. Thus, one might hope that a relatively equal economic and political power distribution endogenously emerges through the players’ equilibrium...
interactions in the long run beginning from noncollegial states. The next lemma answers this question.

**Lemma 4.** If the initial state is noncollegial, the long-run outcome is a permanent oligarchy of two players in equilibrium.

Even if there are multiple players who take the future seriously (perfectly farsighted) and whose power distribution is relatively equal at the beginning, the total wealth and power are concentrated in the hands of exactly two players in the long run. Until being indifferent, the proposer in each period chooses a minimum cost-winning coalition in the sense that the sum of its members’ demands is the smallest among all the decisive coalitions that include the proposer. These proposals in each period reduce the number of players who have proposal and voting rights. This bargaining and coalition building process continues until an oligarchy emerges.

Proposition 1 establishes equilibrium existence and summarizes the previous results.

**Proposition 1.** There exists an equilibrium. In all equilibria, the following statements are true:

(i) If the initial state is dictatorial or tyrannical, the long-run outcome is a permanent tyranny of a single player. The dictator or tyrant in the initial state turns into a permanent tyrant.

(ii) If the initial state is not dictatorial or tyrannical, the long-run outcome is a permanent oligarchy of two players who equally share the entire wealth and power. If the initial state is oligarchic, the oligarchy perpetuates. If the initial state is collegial, the first proposer is always in the permanent oligarchy. If the initial state is noncollegial, the second proposer is always in the permanent oligarchy.

Proposition 1 (i) and (ii) follow from previous Lemmas. The recurring bargaining processes with endogenous status quo and endogenous bargaining power among perfectly farsighted players are eventually stabilized either in tyranny or oligarchy. The long-run outcome depends on the initial distribution of wealth and power among the players. If there is a single player who is unfairly stronger than other players in the initial state as in dictatorial or tyrannical initial states, the long-run outcome is the initially powerful player’s tyranny. On the contrary, if the players begin the bargaining process in a relatively equal position to each other as in noncollegial and collegial states, there is no tyranny in the long run. Instead, a coalition of two players, an oligarchy, emerges in the long run from these states and perpetuates.

Proposition 1 (ii) answers the question who the oligarchs are. Once an oligarchy is in place, the question is trivial because the two oligarchs monopolize proposal rights. Being a proposer is extremely valuable on the transient path to oligarchy. In collegial states, the first proposer is guaranteed to be included in the permanent oligarchy. If the first proposer is the collegium player, the collegium player will be obviously one of the oligarchs since all noncollegium players, once recognized, as oligarchy of oneself and the collegium player. If the first proposer is one of the noncollegium players, the proposer exploits and denies all wealth and power currently possessed by other noncollegium players and forms an oligarchy by cooperating with the collegium player. The advantage of proposers in noncollegial states is more subtle. Although it is more likely that the first proposer is able to be a permanent oligarch, it is not true in general when there are more than three players.\(^6\)

**Example 1.** Consider a noncollegial state \(s = (\frac{4}{13}, \frac{4}{13}, \frac{4}{13}, \frac{1}{13})\). Let \(\mu\) be \(\mu_1(y^1|s) = \mu_1(y^3|s) = \mu_2(y^1|s) = \mu_2(y^2|s) = \mu_3(y^3|s) = \frac{1}{2}\), where \(y^1 = (\frac{1}{2}, \frac{1}{2}, 0, 0), y^2 = (0, \frac{1}{2}, \frac{1}{2}, 0), y^3 = (\frac{1}{2}, 0, \frac{1}{2}, 0)\), without specifying \(\mu_4(s)\). Then for \(i = 1, 2, 3\), \(i\)'s \(T\)-period expected utility from \(s\) satisfies the following inequality:

\[
U^T_i(s) > \frac{1}{T} \left[ \frac{4}{13} \right] + \frac{4(T - 1)}{13} = \frac{4}{13}.
\]

If the first proposer is one of the players 1, 2, 3, the next period is in an oligarchic state, in which the first proposer is a permanent oligarch. However, player 4 cannot take one half of the dollar. Player 4 needs to

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\(^6\) In the three-player game, every player in a noncollegial state can directly propose oligarchic divisions.
guarantee $U^T_i(s)$ to at least two players among players 2, 3, and 4 to pass his proposal, since the decisive coalitions that include player 4 are \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\} and \{1, 2, 3, 4\}. Thus, player 4 gives at least $2U^T_i(s) > \frac{8}{13} > \frac{1}{2}$ to other players and takes the remainder expected utility that is less than a half, which implies player 4 is neither an oligarch nor a collegium player who secures a seat in the permanent oligarchy in the state induced by his proposal.

In Example 1, however, the proposer in the second period can take $\frac{1}{2}$ immediately and secure a position in the permanent oligarchy. If the first proposer were player 1, 2, or 3, the question is trivial because two of these players have already formed an oligarchy after the first period. If player 4 were the first proposer, there remains three players having positive shares in the second period because player 4, who takes the remainder expected utility after satisfying its coalition partners’ demands, chooses exactly two players as coalition partners. In noncolllegial states in which three players have positive shares, any two players can form a decisive coalition. Hence, any proposer in the state induced by player 4’s proposal must be able to assign one half of the dollar to oneself in equilibrium. The following example shows that this intuition on the second proposer’s advantage holds beyond Example 1.

**Example 2.** Let $n = 6$ and consider noncollegial state $s$ in which every player’s status quo share is positive. Suppose that $\sigma = (\mu, A)$ is an equilibrium and $\mu_i(y_i(s)) > 0$, where $y_i \in (0, \frac{1}{2})$ for $i = 1, 2, 3, 4$ and $y_j = 0$ for $j = 5, 6$. That is, player 1’s proposal $y$ in state $s$ is a noncollegial division. Let $U^T_i(s)$ and $U^T_i(y)$ be $i$’s $T$-period expected utilities from $s$ and $y$ in $\sigma$, respectively. Since $\sigma$ is an equilibrium by supposition, we know that player 1 was unable to take one half of the dollar in $s$ and $\sum_{i=2}^{4} U^T_i(s) > \frac{1}{2}$. Player 1 gives precisely the current expected utilities to his coalition partners but no more because he takes the remainder. That is,

$$U^T_i(s) = U^T_i(y) \text{ for } i = 2, 3, 4 \text{ and } U^T_1(y) = 1 - \sum_{i=2}^{4} U^T_i(s).$$

First, I claim that $s_2 + s_3, s_2 + s_4, s_3 + s_4 < \frac{1}{2}$. Otherwise, player 1 could have formed a decisive coalition without one of these players and improved his expected utility from his proposal. Next, I claim that $U^T_1(y) + U^T_2(y) > \frac{1}{2}$ for $i = 2, 3, 4$. Without loss of generality, suppose that $U^T_1(y) + U^T_2(y) \leq \frac{1}{2}$, as a contradiction to the claim. Then $U^T_2(y) + U^T_4(y) = U^T_3(s) + U^T_4(s) \geq \frac{1}{2}$, and $U^T_2(s) + U^T_4(s) + U^T_6(s) < \frac{1}{2}$ by $U^T_i(s) > 0$. Then the proposal $y$ that satisfies the demands of player 1’s coalition partners $\{2, 3, 4\}$ is not optimal for player 1. By $s_3 + s_4 < \frac{1}{2}, \{1, 2, 5, 6\}$ is a decisive coalition in $s$. Also, $U^T_2(s) + U^T_5(s) + U^T_6(s) < \frac{1}{2}$. Thus, player 1 could have proposed a collegial division $y'$ from which his own share $y'_1 = \frac{1}{2}$ and $U^T_1(y') = \frac{1}{2}$ in state $s$, a contradiction to $\sigma$ being an equilibrium. Thus $U^T_1(y) + U^T_1(y) > \frac{1}{2}$ for $i = 2, 3, 4$. Finally, I claim that for $i = 1, 2, 3, 4$, $i$ assigns $\frac{1}{2}$ to oneself in one’s proposal in state $s$, meaning the second proposer secures the position in the future oligarchy. Since $y$ is a noncollegial state, any player can form a decisive coalition with two other players. Thus every player can choose at least one of $\{2, 3\}, \{2, 4\}, \{3, 4\}$ as coalition partners. The sum of demands (expected utilities from $y$) of any two players among $i = 2, 3, 4$ is less than $\frac{1}{2}$ by $U^T_1(y) + U^T_1(y) > \frac{1}{2}$ for $i = 2, 3, 4$. Therefore, all players can take an expected utility of $\frac{1}{2}$ from one’s proposal in $y$, which means that these proposals are either collegial or oligarchic.

In the proof of Proposition 1, I formulate a state game for each state as a variant of the Baron-Ferejohn sequential bargaining game, identify a no-delay symmetric stationary subgame perfect equilibrium in stage-undominated strategies (SSPE) of the state games, and finally show that the collection of the SSPE proposal strategies for all states constitutes an equilibrium proposal strategy profile of the dynamic bargaining game with endogenous political power. Considering the close connection between the canonical Baron-Ferejohn model and the model analyzed in this article, the uniqueness of equilibrium and expected utility is of particular interest. As the simple three-player example demonstrates, there are multiple equilibria and different equilibria may be associated with different expected utility vectors.
Example 3. Consider \( I = \{1, 2, 3\} \) and a noncollegial state \( s = (\frac{1}{3}, \frac{3}{9}, \frac{2}{9}) \). Let \( \mu(s) \) and \( \hat{\mu}(s) \) be two proposal strategies defined at \( s \) such that

\[
\mu_1(y^1 | s) = \mu_2(y^2 | s) = \mu_3(y^3 | s) = \frac{1}{2}
\]

\[
\mu_1(s) = \mu_2(s) = y^1, \quad \hat{\mu}_3(s) = y^3,
\]

where \( y^1 = (\frac{1}{2}, \frac{1}{2}, 0) \), \( y^2 = (\frac{1}{2}, 0, \frac{1}{2}) \), \( y^3 = (0, \frac{1}{2}, \frac{1}{2}) \).

Obviously, both \( \mu(s) \) and \( \hat{\mu}(s) \) are optimal since every proposer takes \( \frac{1}{2} \), the maximum value one can assign to oneself in any noncollegial state. Thus, we can construct equilibrium proposal strategies \( \mu \) and \( \hat{\mu} \), where \( \mu \) assigns \( \mu(s) \) and \( \hat{\mu} \) assigns \( \hat{\mu}(s) \) in \( s \) and \( \mu(s') = \hat{\mu}(s') \) in all other states \( s' \neq s \).

To see that the expected utilities from \( \mu \) and \( \hat{\mu} \) are different, it is enough to compare player 1’s expected utilities \( U^T_1(s) \) and \( \hat{U}^T_1(s) \):

\[
U^T_1(s) = \frac{1}{T} \frac{4}{9} + (T - 1) \left( \frac{4}{9} \cdot \frac{1}{2} + \frac{3}{9} \cdot \frac{1}{2} + \frac{2}{9} \cdot \frac{1}{2} \right) = \frac{13T + 3}{36T},
\]

\[
\hat{U}^T_1(s) = \frac{1}{T} \frac{4}{9} + (T - 1) \left( \frac{4}{9} \cdot \frac{1}{2} + \frac{3}{9} \cdot \frac{1}{2} \right) = \frac{14T + 2}{36T},
\]

showing that \( U^T_1(s) \neq \hat{U}^T_1(s) \) when \( T \to \infty \).

The multiplicity of equilibria stems from the limit endogenously imposed on possible proposals in collegial and noncollegial states. The proposers in collegial and noncollegial states cannot take more than one half of total wealth and total expected utility because other players do not approve dictatorial proposals. Given that, the proposers have many proposals, and an infinite mixture of those proposals, that maximize their own expected utilities. In Example 3, for instance, player 1 can choose player 2 or 3 as a coalition partner with any probabilities and still take one half of the dollar from his proposal, because the expected utilities (demands) of player 2 and 3 are both less than one half regardless of player 1’s proposal plan. Nevertheless, it should be emphasized that in any equilibrium the long-run outcome is either an extreme domination by a single player who monopolizes the entire wealth and power (tyranny) or by two players who equally share the entire wealth and power (oligarchy) excluding all other players from the subsequent bargaining process.

### 4 Imperfect Foresight

The bargaining model of endogenous political power analyzed above has assumed perfect foresight of players. To see the role of farsightedness, suppose that the players are completely myopic on the other extreme. Then each player’s expected utility is simply one’s status quo share, \( U_i(s) = s_i \). Accordingly, myopic player \( i \) approves any proposal \( y \) such that \( y_i \geq s_i \), and the endogenous limit farsighted players impose on the set of possible proposals in collegial and noncollegial states disappears. This myopic attitude of players thus invites a dictator, even if the initial state is not dictatorial. For instance, if the initial state is \( s = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \), the first proposer turns into a dictator immediately, instead of sharing the wealth and power with another player in equilibrium. An interesting question concerns the players who are neither completely myopic nor perfectly farsighted. Would the imperfectly farsighted players be able to prevent transition to dictatorship when the wealth and power distribution is relatively equal at the beginning? In this section, I show that an oligarchy emerges from all initial states even if the players are not perfectly farsighted when the policy space is discrete. First, I introduce the modified game with discounting in the discrete policy space.

The bargaining procedure is analogous to the game introduced in Section 2. A set \( I = \{1, \ldots, n\} \) of players divide a dollar in each period \( t = 1, \ldots, \infty \). Each period begins in state \( s \), where \( s \) represents the
For any Proposition 2. model, we have the following result: the probability and voting weights are adjusted according to the bargaining outcome in the immediate previous period. The status quo proposal is passed and becomes a new status quo if it obtains more than half of the total voting weights; the status quo remains in effect otherwise. Before the beginning of the next period, the players recognize that the bargaining outcome and voting weights are adjusted according to the bargaining outcome in the immediate previous period. The players discount the future by a common factor $\delta \in (0, 1)$.

Let $\sigma = \{(\mu_i, A_i)\}_{i \in I}$ be any symmetric stationary Markov strategies. Let $I_{A_i}(y) = 1$ if proposal $y$ is in the social acceptance set in $s$ and 0 otherwise. The players’ continuation value $v^\tau(s)$ and expected utility $U^\sigma(s)$ from state $s$ given $\sigma$ are given by, for every $i \in I$,

$$v^\tau_i(s) = \sum_{y \in \cal{Y}} s_j \sum_{y: \mu_j(y) \geq 0} \mu_j(y) s \{ U^\tau_i(y) I_{A_i}(y) + U^\tau_i(s)(1 - I_{A_i}(y)) \}$$

and

$$U^\tau_i(s) = (1 - \delta) s_i + \delta v^\tau_i(s).$$

The equilibrium notion is as defined in Definition 1 with $x \geq y \iff U_i(x) \geq U_i(y)$. In this modified model, we have the following result:

**Proposition 2.** For any $\tau$, there exist $\bar{\delta}^\tau < 1$ such that there is an equilibrium for all $\delta > \bar{\delta}^\tau$. In all equilibria, 
(i) none of the players accept other players’ dictatorship in collegial and noncollegial states; 
(ii) the long-run outcome is tyranny of a single player if the initial state is dictatorial or tyrannical; and 
(iii) the long-run outcome is oligarchy of two players if the initial state is not dictatorial or tyrannical. 
Furthermore, $\bar{\delta}^\tau \to 1$ as $\tau \to \infty$.

Sufficiently farsighted players ($\delta > \bar{\delta}^\tau$), even if not perfectly farsighted, never invite a tyrant in equilibrium unless there is a dictator or tyrant in the initial state, given that they cannot divide the dollar in a continuous manner. Instead, an oligarchy of two players emerges and perpetuates in the long run. Also, the necessary level of farsightedness to enable oligarchy instead of tyranny increases as the players are able to divide the dollar into a smaller unit (increase in $\tau$). When $\tau \to \infty$, the set of proposals $X'$ approximates the continuous policy space $X$, and in this case, the players needs to be almost perfectly farsighted to prevent tyranny.

To get the intuition behind the relationship between $\tau$ and $\bar{\delta}^\tau$, consider the following example.

**Example 4.** Suppose that $n = 3$ and $\tau = 12$. Let $s$ be a collegial state with $s_c = \frac{1}{2}$ and $s_i = \frac{1}{12}$. If all noncollegial players do not accept any of the collegium player’s collegial proposals, the collegium player proposes the status quo and $i$ proposes the oligarchic division in which $c$ and $i$ equally share the dollar. Thus, player $i$’s expected utility from $s$ in such proposal and voting plans is $U_i(s) = \frac{1 - \delta}{12} + \delta \left( \frac{5U_c(s)}{2} + \frac{1}{12} \cdot \frac{1}{2} \right) = \frac{1}{12}$. Let $y \in X_D$ be a possible collegial proposal by player $c$ such that $y_c = \frac{7}{12}$ and $y_i = \frac{5}{12}$. If $i$ and $c$ agrees to move onto $y$, player $c$ will propose his tyrannical division and player $i$ will propose the status quo in state $y$ in equilibrium. Then player $i$’s expected utility from $y$ is $U_i(y) = \frac{5U_c(y)}{12} + \frac{5U_c(y)}{12} = \frac{5(1 - \delta)}{12}$. Player $i$’s voting plan not to accept player $c$’s collegial proposal can be supported in equilibrium only if $i$ is more farsighted than the threshold level $\delta = \frac{48}{52} \approx 0.87$, which is from $\frac{1}{12} > \frac{5(1 - \delta)}{12}$. 

Now suppose that $\tau = 24$ and consider player $c$’s collegial state $s'$ in which $s'_c = \frac{1}{24}$ and $s'_i = \frac{1}{24}$. If player $c$ cannot propose his dictatorship, player $i$’s expected utility from $s'$ is $U_i(s') = s'_i = \frac{1}{24}$. Let $y'$ be player
c’s dictatorial division most favorable to player $i$: $y'_c = \frac{13}{24}$ and $y'_i = \frac{11}{24}$. We can compute player $i$'s expected utility from $y'$ in the same way from above that $U_i(y') = \frac{11(1-\delta)}{24-11\delta}$, and the threshold is $\delta^* = \frac{240}{253} \approx 0.95$. ■

As seen in the example, $\tau$ determines both the lowest level of welfare in non-dictatorship and the ability of an aspiring dictator—the collegium player in the above example—to reward the supporters of his dictatorship. When $\tau$ gets larger, the minimum positive status quo share gets smaller in non-dictatorship and the reward a dictator can provide gets larger. It makes it harder for players to resist other players’ dictatorial plans in some states. Accordingly, the players needs to be more farsighted so that the value of even a slight chance of being an oligarch in the future overwhelms the value of receiving a larger short-term payoff under the risk of being permanently deprived in another player’s dictatorship.

5 Conclusion

I have presented a dynamic model of endogenous political power. Specifically, I have analyzed an infinite horizon divide-the-dollar game with an endogenously determined status quo among perfectly farsighted players. In the model, the proposal and voting power of players also evolve endogenously through the bargaining process. Given that the players are farsighted, the long-run outcome depends on the initial state. If any single player is overwhelmingly wealthy and powerful at the beginning, this player eventually takes all resources and power available to a society and becomes a tyrant. This tyranny then remains forever. Anticipating the behavior of these powerful individuals, the players endowed with relatively equal wealth and power in the initial state do not exchange their chance of long-term survival for short-term gains a dictator might provide. In noncollegial and collegial states farsighted players do not approve other players’ dictatorial plans, which prevents the advent of dictatorship and tyranny. Instead of tyranny, these players’ repeated interactions beginning from a relatively equal initial state lead to the emergence of an oligarchy.

Comparing perfectly farsighted players with completely myopic players, we can also infer the importance of farsightedness. Completely myopic players induce a tyranny at the end of the day from almost all initial states including noncollegial states. If three players have one third of the dollar initially, for instance, the first proposer always becomes a dictator since other players agree to any proposal as long as one’s share remains one third. This dictator then turns into a tyrant as soon as he becomes a proposer next time. As such, farsightedness is crucial in preventing transition to dictatorship. Admittedly, it is hard to expect perfect foresight for individuals or groups within a society. I have shown in Section 4 that sufficiently farsighted players behave in the same way with their perfectly farsighted counterparts, given that the dollar is divided in discrete units. That is, sufficiently farsighted players, even if they are not perfectly farsighted, prevent the onset of tyranny and form an oligarchy of two players in the long run unless the initial state is dictatorial. The level of farsightedness necessary to deter dictatorship depends on the size of the unit $\frac{1}{\tau}$. A larger $\tau$ requires a higher level of farsightedness, and the model with perfectly farsighted players and continuous policy space can be approximated by letting $\tau \to \infty$. The grid size $\tau$ can be interpreted as the level of sophistication of the players or the technological constraints the players face in a given bargaining environment. Consider, for instance, a number of political groups that negotiate on a new set of political institutions such as electoral systems, campaign finance laws, eligibility of candidates and voters, control over police and military, and access to media, to list a few. The arrangement of institutions is the resource to divide and the bargaining outcome determines each group’s future political power. If the political elites can manipulate the institutional settings to a very micro-level, a group that aspires to dominate the political process can easily find a way to reward those who are relatively vulnerable and willing to support a dictatorship if the instantaneous payoff is large enough. Thus the sophisticate elites under weak institutional constraints are more likely succeed in transition to dictatorship for a given level of farsightedness.

It is worthwhile to emphasize that this model predicts that farsighted players successfully resist to dictatorship. However, the alternative is a perpetuating oligarchy. Both tyranny and oligarchy exhibit severe
concentration of power. Most players, at the end of the day, have no opportunity to advance their interests because they have no proposal power, and no powerful individuals listen to them because they have no voting power. In other words, all but a single player in a tyranny and all but two players in an oligarchy end up being politically irrelevant. Returning to the examples of regime change, we may perceive the initial stage of democratization as an unpredicted shock to a society that disturbs existing economic and political power configuration. My model predicts that the society returns back to dictatorship, if the political players active during the transition are not patient enough to endure short-term hardships and willing to cooperate with a future dictator. If the political actors are farsighted enough, the transition to democracy succeeds but ends up with an perpetual oligarchy of two large parties or factions. This result is consonant with the earlier cautions that democratic projects may serve to legitimately protect the privileged or end up merely replacing one elite group with another. Przeworski (1992) wrote:

Note that partners to such pacts extract private benefits from democracy and that they protect their rents by excluding outsiders from competition. Democracy turns out to be a private project of leaders of some political parties and corporatist associations, and oligopoly in which leaders of some organizations collude to prevent outsiders from entering (p.124).

The oligarchy in this model refers to a very specific form of coalition— a coalition of two players sharing total resources and power equally. The perpetual oligarchy of two players is due to the majoritarian structure of the game. Any proposal needs support of a majority of total voting weights. If alternation of existing wealth and power distribution requires a supermajority instead of a simple majority, which is the case of weighted $q$-rules, the size and composition of the permanent oligarchy change. For instance, suppose that a proposal needs more than two thirds of total voting weights to pass. Then the states in which any three players equally share the resources and power continue forever, because every player of the three has a veto. As $q$ approaches one, almost all states are stable in the long run.

Finally I suggest directions for fruitful future research. First, one might consider a model in which the number of players that enjoy a certain level of wealth and power affects the productivity of the economy—the amount of total available resources. In this case players will need to consider not only one’s own share out of the current resources and the stability of one’s political position but also the change of the total available resources in the next period. Forming an oligarchy still ensures the oligarchs stable economic and political status relative to most of the players but may not maximize long-term utilities by reducing the productivity of the economy. Second, existing bargaining models show that risk-aversion is one of the primary sources that induce players to compromise and form a larger stable coalition. My model assumes risk-neutral players, and the proofs for the results heavily utilize this feature. Investigating the possibility of a larger stable coalition than the oligarchy of two players and the relationship between the degree of risk-aversion and the necessary level of farsightedness to maintain a certain size of stable coalition will be an interesting future research agenda.\(^7\)

\(^7\)I conjecture that a larger stable coalition may be formed if the players are sufficiently risk averse. To give an intuition behind the conjecture, I provide an example using a discrete policy space and four sufficiently farsighted players, where sufficient foresight guarantees no transition to dictatorship from non-dictatorial states. Let the instantaneous utility be given by $u_i(s) = \sqrt{\epsilon}$ and consider a strategy in which all players propose the status quo $s$ if $s = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. These proposals can be supported in equilibrium because the players can propose neither an equal division among three players nor a collegial division in which the two noncollegium players’ shares are one fourth. In the given proposal plans, $U_i(s) = \frac{1}{3}$ for every player. For $y = \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right)$, $\lim_{\epsilon \rightarrow 0} U_2(y) = \frac{1}{2\epsilon} + \frac{\sqrt{3}}{2\epsilon^2} = \frac{\sqrt{27}}{2} < \frac{1}{3} = U_2(s)$. Also for $z = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $\lim_{\epsilon \rightarrow 0} U_2(z) = \frac{1}{2\epsilon} + \delta\left(\frac{1}{2} U_2(z) + \frac{1}{4\epsilon}\right) = \frac{\sqrt{27}}{4} < \frac{1}{3} = U_2(s)$. One can easily see that $y$ and $z$ are the best possible offers player 1 can make to player 2 and 3 in exchange for their supports among all noncollegial and collegial divisions. Thus, player 1, likewise all other players, cannot successfully propose an allocation other than the status quo.
References


Stiglitz, Joseph E. 2011. “Of the 1%, by the 1%, for the 1%.” Vanity Fair.

URL: http://www.vanityfair.com/society/features/2011/05/top-one-percent-201105
Appendix A: Formal Proofs

Proof of Lemma 1. (i) It is obvious from the text. (ii) Consider \( s \in S_{D_1} \) and let \( \sigma(s) \) be the proposals and acceptance sets in \( s \) as described in (ii). Player 1’s tyrannical proposal is optimal and in the social acceptance set by (i). Given \( \sigma(s) \), player 1’s \( T \)-period expected utility is

\[
v_1^T(s) = s_1 + s_1(1 - s_1)
\]

\[
U_1^T(s) = \frac{s_1}{2}(1 + (1 - s_1))
\]

\[
v_2^T(s) = \frac{2s_1 + (1 - s_1)(s_1 + v_1^T(s))}{2} = s_1 + s_1(1 - s_1) + \frac{1}{2}s_1(1 - s_1)^2
\]

\[
U_2^T(s) = \frac{s_1}{3}(3 + 2(1 - s_1) + (1 - s_1)^2)
\]

\[
U_1^0(s) = \frac{s_1}{T} \sum_{t=1}^{T} (1 - s_1)^{T-t},
\]

which is increasing in \( s_1 \). The dictator is thus indifferent between \( s \) and any other proposal that guarantees his status quo share \( s_1 \). Hence, proposal \( y \) by any nondictator \( i \) such that \( y_1 = s_1 \) prescribed in \( \sigma(s) \) is in the social acceptance set. To see that nondictator \( i \)'s proposal \( y \) is optimal and unique, it suffices to state that

\[ U_i^T(s) = U_i^T(y) \quad \text{and} \quad U_i^T(y) = 1 - U_i^T(s). \]

(iii) Let \( s \) be an oligarchic state in which players 1 and 2 are in the oligarchy and \( \sigma(s) \) be the proposals and acceptance sets in \( s \) as described. For \( i \in \{1, 2\} \), \( U_i^T(s) = \lim_{T \to \infty} U_i^T(s) = \frac{1}{2} \) in \( \sigma(s) \). Since \( \lim_{T \to \infty} U_i^T(y) = 0 < \frac{1}{2} \) for all \( y \in X_{D_2} \), player 1 does not accept player 2’s dictatorial proposals, and vice versa. Also, player 1 does not assign a positive share to other players \( j \neq 1, 2 \) in his proposal \( y \) since it must be \( U_1^T(y) = U_1^T(s) \) and \( y_j > 0 \) implies that \( U_1^T(y) < U_1^T(s) = \frac{1}{2} \). The same holds for player 2. \( \blacksquare \)

I prove Lemma 3 first, and then Lemma 2.

Proof of Lemma 3. Suppose on the contrary that there is a noncollegial state \( s \) in which at least one player proposes and passes his dictatorial division \( y \) with support of a decisive coalition. Without loss of generality, let player 1 be the proposer and player 2 belong to the decisive coalition that votes for \( y \in X_{D_1} \). Then \( v_1^{T-1}(s) \) is at least \( s_1 U_1^{T-1}(y) \). Player 1 is recognized with probability \( s_1 \) and proposes \( y \) that gives \( U_1^{T-1}(y) \) thereafter. Since \( s \) is noncollegial, player 2 can secure at least \( U_1^T(s) \), when recognized, by proposing a permutation of \( s \) in which only \( s_1 \) and \( s_2 \) are exchanged, which means \( v_2^{T-1} > s_2 U_1^T(s) \). Using a dictator’s expected utility in (9), then, we have

\[
v_1^{T-1}(s) \geq s_1 U_1^{T-1}(y) = \frac{s_1[y_1(T - 1) - (1 - y_1)(1 - (1 - y_1)^{T-1})]}{y_1(T - 1)}
\]

\[
U_1^T(s) > \frac{T - 1}{T} v_1^{T-1}(s) > \frac{s_1[y_1(T - 1) - (1 - y_1)(1 - (1 - y_1)^{T-1})]}{y_1 T}
\]

\[
v_2^{T-1}(s) > s_2 U_1^T(s) > \frac{s_1 s_2[y_1(T - 1) - (1 - y_1)(1 - (1 - y_1)^{T-1})]}{y_1 T}
\]

\[
U_2^T(s) > \frac{T - 1}{T} v_2^{T-1}(s) > \frac{s_1 s_2[y_1(T - 1)^2 - (1 - y_1)(1 - (1 - y_1)^{T-1})(T - 1)]}{y_1 T^2}
\]
Finally,
\[ \lim_{T \to \infty} U^T_2(s) \geq s_1 s_2 y_1 > 0 = \lim_{T \to \infty} U^T_2(y) \quad \text{and} \quad \lim_{T \to \infty} T[U^T_2(y) - U^T_2(s)] < 0, \]

implying \( y \notin A_2(s) \), a contradiction to the supposition that player 2 accepts \( y \). ■

**Proof of Lemma 2.** Consider collegial state \( s \in S_{COL_1} \) and a noncollegium player \( i \). Denote the proposals and voting plans described in Lemma 2 by \( \sigma(s) \). First, I show that the collegium player 1’s proposal \( s \) is the unique optimal proposal. Given \( \sigma(s) \), \( U^T_1(s) = \frac{1}{T} \) and

\[ u^1_1(s) = \frac{s_1 + s_{\frac{1}{2}}}{1} = s_i \]
\[ u^2_1(s) = \frac{\frac{1}{2}[s_1 + u^1_1(s)]}{1} = \frac{2s_i}{2} = s_i \]
\[ u^{T-1}_1(s) = \frac{s_1 + (T - 2)u^{T-2}_1(s)}{T - 1} = s_i, \]

accordingly,

\[ U^T_1(s) = \frac{s_1 + (T - 1)u^{T-1}_1(s)}{T} = \lim_{T \to \infty} U^T_1(s) = s_i > 0. \] (13)

By \( \lim_{T \to \infty} U^T_1(z) = 0 \) for any dictatorial allocation \( z \in X_{D_1} \), noncollegium player \( i \) does not accept player 1’s dictatorial proposal if proposed. Since there is no transition from collegium and noncollegium states to dictatorial states in \( \sigma(s) \) and by Lemma 3, the social acceptance set contains a subset of player 1’s collegial states and oligarchic states, all of which gives an expected utility \( \frac{1}{2} \) to player 1. Player 1, being indifferent between all proposals in the social acceptance set, then proposes the status quo \( s \).

Player 1 obviously accepts all noncollegium players’ oligarchic proposals because these oligarchic proposals and the status quo give the same expected utility \( \frac{1}{2} \) to player 1. To see that the noncollegium players’ oligarchic proposals, say \( y \), are unique optimal proposals, it suffices to state that \( U^T_1(s) = U^T_1(y) \) and \( U^T_1(y) = 1 - U^T_1(y) \). Any other proposal gives a smaller expected utility to either player 1 or player \( i \) or both. ■

**Proof of Lemma 4.** Suppose that \( s \in X_{NC} \), and let \( m \leq n \) players have positive status quo shares. Align the indices of players so that \( s_1 \geq \ldots \geq s_m \geq \ldots \geq s_n \). If \( s_i = s_j \) for all \( i, j \leq m \), it is easy to see that every \( i \leq m \) can take \( \frac{1}{2} \) in one’s proposal by symmetry and the number of other players’ votes necessary to pass a proposal. If \( s_i \neq s_j \) for some \( i, j \leq m \), then \( s_i > s_m \). Also there is a player \( h \geq 2 \) with the smallest index such that \( \sum_{i \leq h} s_i \geq \frac{1}{2} \). If \( \sum_{i \leq h} s_i > \frac{1}{2} \), player 1 (in fact every \( i \leq h \)) has at least two disjoint sets of coalition partners to make decisive coalitions: \( C_1 = \{2, \ldots, h\} \) and \( C_2 = \{h + 1, \ldots, m\} \). Since \( \sum_{i \leq j} U^T_1(s) = 1 \), either \( \sum_{i \in C_1} U^T_1(s) \leq \frac{1}{2} \) or \( \sum_{i \in C_2} U^T_1(s) \leq \frac{1}{2} \) or both regardless of \( T \). Thus, player 1 needs to pay less than or equal to \( \frac{1}{2} \) total expected utilities to other players to pass his proposal, which enables player 1 to be either a collegium player in a collegial state or an oligarch in an oligarchic state in the next period. If \( \sum_{i \leq h} s_i = \frac{1}{2} \), player 1 can again form decisive coalitions with at least two disjoint sets of players: \( C_1 = \{2, \ldots, h, m\} \) and \( C_2 = \{h + 1, \ldots, m - 1\} \), and the same argument above applies. Accordingly, any noncollegial state turns to a collegial or oligarchic state between periods with positive probability, and such a process results in an oligarchy in the long run. ■

I prove Proposition 2 using the following lemmas.

**Lemma 5.** In any noncollegial state \( s \) and for any \( i \) with \( s_i > 0 \), \( U_i(s) \geq \frac{3 - \delta}{3(\pi - \delta)} \).
Proof of Lemma 5. Let $s \in S_{NC}$ and align the players so that $s_1 \geq \ldots, s_n$. Since $s$ is noncollegial, there is a player $h \geq 2$ such that $\sum_{j \leq h} s_j \geq \frac{1}{\tau}$. Let $i$ be the players with index $h < i$ and $s_i = \frac{1}{\tau}$. Let $k$ be the player indexed $k \leq h$ with $U_k(s) \leq U_i(s)$, $\forall j \leq h$. Player $i$ has at least two sets of decisive coalition partners to form when recognized to propose: $C_1 = \{1, \ldots, h\}$ and $C_2 = \{k, h + 1, \ldots, n\} \setminus \{i\}$. If $i$’s proposal is optimal, $i$ forms a coalition with the partners whose sum of demands in terms of expected utility is smaller between $C_1$ and $C_2$. Let $U = \sum_{j \in C_1} U_j(s)$ and $\bar{U} = \sum_{j \in C_2} U_j(s)$. The sum of all players’ expected utilities is 1, thus, $\bar{U} = 1 - U + U_k(s) - U_i(s)$. Since $k$’s expected utility is the smallest among the players $j \leq h$, $U_k(s) \leq \frac{U}{h}$. Player $i$ chooses $C_1$ as coalition partners if $U > \bar{U}$ and $C_2$ if $U < \bar{U}$. Thus, the largest demands of $i$’s chosen coalition partners occur when $U = \bar{U}$, which is
\[
\max_{h \geq 2}[U = \bar{U}] \iff \max_{h \geq 2}[U = 1 - U + \frac{U}{h} - U_i(s) = \frac{h(1 - U_i(s))}{2h - 1}] = \frac{2(1 - U_i(s))}{3}.
\]
Then a lower bound of $i$’s expected utility from $s$ is
\[
U_i(s) \geq (1 - \delta)s_i + \delta s_i(1 - U) \\
\geq \frac{(3 - 2\delta)s_i}{3 - s \delta s_i} = \frac{3 - 2\delta}{3 - 2\delta}
\]
(14)
Note that there is a possibility that (14) is not the greatest lower bound depending on specific equilibrium in use. But the expected utility of any player who has a positive status quo share in any noncollegial state is at least (14) in equilibrium because it is derived assuming that no other player chooses $i$ as a coalition partner and $i$’s status quo share is the smallest unit $\frac{1}{\tau}$. ■

Lemma 6. If there is no transition to dictatorship from collegial or noncollegial states,

\[
U_i(s) = \begin{cases} 
1 & \text{if and only if } s_i = 1 \\
\geq \frac{\tau + 2}{2\tau - (r - 2)\delta} & \text{if } s \in X_{Di}, \\
\geq \frac{\tau + 2}{\tau - (r - 2)\delta} & \text{if } s \in X_{COL,i}, \\
\geq \frac{3 - 2\delta}{2\tau - 2\delta} & \text{if } s \in X_{NC}, \\
\leq \frac{(1 - \delta)(r - 2)}{2\tau - 2\delta} & \text{if } s \in X_{D,i},
\end{cases}
\]
in equilibrium.

Proof of Lemma 6. First notice that Lemma 1 directly applies regardless of $\delta$. (i) In tyrannical states, the tyrant proposes the status quo, which makes $U_i(s) = 1$. Obviously the expected utility from all other states are less than one, because the instantaneous utility from all other states is less than one. (ii) In dictatorial states, the dictator proposes his tyranny because he can pass any proposal he wants. Non-dictators satisfy the dictator by giving the same status quo share to the dictator and take the remainder. From this, we have $U_i(s) = s_i(1 - \delta) + \delta[s_i + (1 - s_i)U_i(s)] = \frac{s_i}{1 - s_i} \geq \frac{\tau + 2}{2\tau - (r - 2)\delta}$ when $i$ is the dictator in $s$ and having $\frac{1}{2} + \frac{1}{\tau}$. Relatively, when $i$ is a non-dictator in another player’s dictatorial state, the maximum expected utility is the total expected utility 1 minus the dictator’s minimum possible expected utility. That is $U_i(s) = 1 - \frac{(1 - \delta)(r - 2)}{2\tau - 2\delta}$ when $i$ is a non-dictator in another player’s dictatorship. (iii) In oligarchic states, each oligarch obviously does not approve each other’s dictatorship and proposes the status quo, which gives an expected utility of $\frac{1}{2}$. (iv) Under the assumption that there is no transition to dictatorship from collegial or noncollegial states, the expected utilities of every player in any noncollegial state is strictly less than one half. Thus the collegium player does not accept any noncollegial proposals, and the collegium player cannot propose his dictatorship because there is no transition to dictatorship from collegial states by supposition,
which means his expected utility is exactly one half and becomes indifferent between all divisions in which his share is \( \frac{1}{2} \). Due to this indifference over all socially acceptable proposals, the collegium player proposes the status quo. Given the indifference of the collegium player over all allocations that gives one half of the dollar to oneself, each noncollegium player’s optimal proposal is an oligarchy in which the collegium player and oneself share the entire dollar equally. Given this, a noncollegium player’s expected utility is \( U_i(s) = s_i(1 - \delta) + \delta(U(s) + s_i) = s_i \geq \frac{1}{\tau} \), since \( s_i > 0 \) means that \( s_i \geq \frac{1}{\tau} \). (v) Finally, if \( s \) is a noncollegial state, \( U_i(s) \geq \frac{3 - 2\delta}{3\tau - 2\delta} \) by Lemma 5. ■

**Lemma 7.** There exists \( \delta^T < 1 \) such that for all \( \delta > \delta^T \), none of the players having a positive status quo share accept other players’ dictatorial proposals.

**Proof of Lemma 7.** We need to find the threshold \( \delta^T \) that satisfies

\[
\min_{s \in X_{COL}} U_i(s) > \max_{y \in X_{D,i}} U_i(y) \quad \text{and} \quad \min_{s \in X_{NC}} U_i(s) > \max_{y \in X_{D,i}} U_i(y),
\]

which is reduced to

\[
\min_{x \in X_{COL}} U_i(x) = \frac{1}{\tau} > \frac{(1 - \delta)(\tau - 2)}{2\tau - (\tau - 2)\delta} \max_{y \in X_{D,i}} U_i(y) \quad \text{and} \quad \min_{x \in X_{NC}} U_i(x) = \frac{3 - 2\delta}{3\tau - 2\delta} > \frac{(1 - \delta)(\tau - 2)}{2\tau - (\tau - 2)\delta} \max_{y \in X_{D,i}} U_i(y).
\]

Solving the two inequalities, we have \( \delta > \frac{\tau(\tau - 4)}{(\tau - 1)(\tau - 2)} \) and \( \delta > \frac{3\tau(\tau - 4)}{3\tau^2 - 11\tau + 2} \). In other words, none of the non-collegium players in collegial states and none of the players having a positive status quo share in noncollegial states accept other players’ dictatorial proposals if \( \delta > \max\{\frac{\tau(\tau - 4)}{(\tau - 1)(\tau - 2)}, \frac{3\tau(\tau - 4)}{3\tau^2 - 11\tau + 2}\} = \frac{3\tau(\tau - 4)}{3\tau^2 - 11\tau + 2} \). Let \( \delta^T = \frac{3\tau(\tau - 4)}{3\tau^2 - 11\tau + 2} \). It is easy to see that \( \delta^T < 1 \) for any \( \tau \) by \( \frac{3\tau(\tau - 4)}{3\tau^2 - 11\tau + 2} = \frac{3\tau(\tau - 4)}{3\tau(\tau - 4) + (\tau + 2)} \) ■

**Proof of Proposition 2.** Equilibrium exists since the set of players, possible proposals (pure actions) and the states are finite. For statement (i), see Lemma 7. By Lemma 7, we know that for any \( \tau \), there is \( \delta^T < 1 \) such that for all \( \delta > \delta^T \), there is no transition to dictatorship from collegial or noncollegial states. In tyrannical, dictatorial, oligarchic, and collegial states, the equilibrium proposals and acceptance sets are identical to those given in Lemmas 1 and 2. Thus, we know that the long-run outcome is tyranny if the initial state is tyrannical or dictatorial; and oligarchy if the initial state is oligarchic or collegial. Applying the same logic with Lemma 4, we see that an oligarchy emerges from all noncollegial states in the long run. Finally,

\[
\lim_{\tau \to \infty} \delta^T = \lim_{\tau \to \infty} \frac{3\tau(\tau - 4)}{3\tau^2 - 11\tau + 2} = 1.
\]

■
Appendix B. Proof of Proposition 1: equilibrium existence

Equilibrium proposals in tyrannical, dictatorial, oligarchic and collegial states are in previous lemmas. Below I formulate state game \( G(s) \) for each noncollegial state \( s \) in which all players’ proposal power and voting weights are fixed, and I seek for no-delay symmetric stationary subgame perfect equilibria in stage-undominated-voting strategies (SSPE) for each state game. I then show that the collection of SSPE proposals in each noncollegial state and the proposals in tyrannical, dictatorial, oligarchic and collegial states found in Lemmas 1 and 2 constitute equilibrium proposal strategy \( \mu^* \) in the whole game. Denote the original game introduced in Section 2 by \( \Gamma \).

State game \( G(s) \) Partition \( S_{NC} \) into two disjoint sets \( S_{NC}^e \) and \( S_{NC}^{ne} \), where \( S_{NC}^e \) is the set of noncollegial states in which every player having a positive status quo share can form a decisive coalition made of two players, and \( S_{NC}^{ne} = S_{NC} \setminus S_{NC}^e \). For each state \( s \in S_{NC} \) and \( i \in I \), let \( L^s_i \) be the set of decisive coalitions that include \( i \). Let \( \lambda^s_i = |L^s_i| \) and index the decisive coalitions in \( L^s_i \) by \( L^s_{i\phi} \) for \( \phi = 1, \ldots, \lambda^s_i \).

For each \( s \in S_{NC} \), formulate state game \( G(s) \) as follows: A set of players, \( I \), divide a dollar for time \( t = 1, \ldots, \infty \). In the initial period, a player is randomly selected as a proposer according to recognition rule \( s \) and proposes a division of the dollar, \( V \in \mathcal{V} \), where \( \mathcal{V} = \{V' \in \mathbb{R}^I | \sum_{i \in I} V'_i = 1, 0 \leq V'_i \leq \frac{1}{2}, \forall i \in I \} \) is the set of all feasible proposals. Note that no one can take more than half of the dollar by the definition of \( \mathcal{V} \). Observing proposal \( V \), every player simultaneously votes to accept or reject it. Player \( i \)'s voting weight is \( s_i \), and the proposal is passed if it obtains more than half of the total voting weights in support. If \( V \) is passed, each player \( i \) receives \( V_i \) in every period henceforth. Otherwise, each player \( i \) receives \( s_i \) for that period, and the same bargaining procedure is repeated in the next period until there is an agreement.

Strategies and preferences The players employ symmetric stationary strategies in \( G(s) \). That is, the players’ proposal and voting strategies remain the same every period, and no two players behave or are treated differently merely because of their names. Let \( \pi^s_i \) and \( a^s_i \) be player \( i \)'s proposal and voting strategies, respectively, where \( \pi^s_i(V') \) is the probability \( i \) proposes \( V' \) and \( a^s_i(V') \in \{ \text{Accept, Reject} \} \) indicates whether \( i \) will accept or reject \( V' \) for \( V' \in \mathcal{V} \). Assume that the support of \( \pi^s_i \) is finite for all \( i \) and \( s \in S_{NC} \). Denote the set of all possible \( \pi^s_i \) by \( \Pi^s_i \) and the set of all possible proposal strategy profiles by \( \Pi^s \). Let \( (\pi^s, a^s) \) be any symmetric stationary strategy profile.

The players do not discount the future flows of incomes. For any finite \( T \), if proposal \( V \) is passed in period \( k < T \), player \( i \)'s long-term utility is simply the \( T \)-period average income \( s_i(\sum_{t=0}^{T-1} V_i \cdot I(a^s, V) + \frac{T}{T-1} \sum_{t=0}^{T-2} \pi^s_i(V) \cdot \text{I}(a^s, V)) \). If the players agree on \( V \) in the first period, \( i \)'s average income is \( V_i \) regardless of \( T \). For any \( (\pi^s, a^s) \), define a \( (T - 1) \)-period continuation value \( \bar{r}^{s,T-1}(\pi^s, a^s) \) and a \( T \)-period reservation value \( r^{s,T}(\pi^s, a^s) \) by, for every \( i \),

\[
\bar{r}^{s,T-1}_i(\pi^s, a^s) = \sum_{j \in I} s_j \sum_{V : \pi^s_j(V) > 0} \pi^s_j(V) \left[ V_i \cdot I(a^s, V) + \frac{s_j + (T - 2)\bar{r}^{s,T-2}_i(\pi^s, a^s)}{T - 1} \cdot [1 - I(a^s, V)] \right]
\]

and

\[
r^{s,T}_i(\pi^s, a^s) = \frac{s_i + (T - 1)\bar{r}^{s,T-1}_i(\pi^s, a^s)}{T},
\]

where \( I(a^s, V) = 1 \) if all members of a decisive coalition support \( V \) in \( a^s \) and 0 otherwise. Given \( (\pi^s, a^s) \), player \( i \) expects to receive \( r^{s,T}_i(\pi^s, a^s) \) in case the first-period proposal is rejected: \( s_i \) for that period and \( \bar{r}^{s,T-1}_i(\pi^s, a^s) \) in each of the next \( T - 1 \) periods, whose sum is averaged over \( T \) periods. Note that for all \( T \) and in any \( (\pi^s, a^s) \), \( \sum_i r^{s,T}_i(\pi^s, a^s) = 1 \).

Since \( T \to \infty \), let player \( i \) prefer accepting proposal \( V' \) to rejecting it in \( (\pi^s, a^s) \) if and only if

\[
\lim_{T \to \infty} T[V'_i - r^{s,T}_i(\pi^s, a^s)] \geq 0.
\]

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**Equilibrium notion** I seek for no-delay symmetric stationary subgame perfect equilibria in stage-undominated voting strategies (henceforth SSPE) of the state game $G(s)$. Every player votes for or against a proposal sincerely, and every proposer makes a proposal that will be supported by at least one decisive coalition and maximizes one’s own average utility. Formally, $(\pi^a^s^r, a^s^r)$ is an SSPE if and only if, for every $i$,

$$a_i^s (V') = \{\text{Accept}\} \iff \lim_{T \to \infty} T[V'_i - r_i^s (V') \geq 0 \}
$$

and

$$\pi_i^s (V') > 0 \Rightarrow V' \in \arg \max_{V \in V} \{V_i | I(a^s^r, V) = 1\}
$$

where

$$r_i^s = \frac{s_i + (T - 1) \sum_{j \in I} s_j \sum_{V, \pi_i^s (V) > 0} \pi_i^s (V) V_i}{T}.
$$

**SSPE of G(s) in s ∈ SNc** Consider any $s \in S_N^c$. For each $i$, let $\Delta L_i^s = \bar{\Pi}_i^s$ be the set of probability distributions over the set of decisive coalitions to which $i$ belong and $\bar{\pi}_i^s \in \bar{\Pi}_i^s$ be $i$’s coalition formation strategy in $s$, where $\bar{\pi}_i^s (L_{i\phi})$ is the probability $i$ assigns to $L_{i\phi} \in L_i^s$. Let $\mathcal{R} = \{r \in \mathbb{R}^n | \sum_i r_i = 1, 0 \leq r_i \leq \frac{1}{2} \text{ for all } i \in I\}$ and fix any $r \in \mathcal{R}$. Given $r$, define $V_i(r)$ for each $i$ and $i$’s decisive coalition $L_{i\phi} \in L_i^s$ by

$$V_i^s (r) = \min \left\{ \frac{1}{2}, 1 - \frac{1}{\sum_{j \in I} \sum_{r_j \in \mathcal{R}} r_j} \right\}
$$

$$V_i^s (r) = r_j + \frac{1}{\sum_{j \not\in \mathcal{R}} \sum_{r_j \in \mathcal{R}} r_j} \text{ for } j \in L_{i\phi} \backslash \{i\}
$$

$$V_i^s (r) = 0 \text{ for } k \not\in L_{i\phi}.
$$

Notice that for all $i, \phi = 1, \ldots, L_i^s$ and $r \in \mathcal{R}$, $\sum_{j \in I} V_i^s (r) = 1$. Define a function $f_i^s : \bar{\Pi}_i^s \times \mathcal{R} \to \mathcal{R}$ by

$$f_i^s (\bar{\pi}_i^s, r) = \sum_{j \in I} \pi_j^s (L_{i\phi}) V_i^s (r) \text{ and a correspondence } B_i^s : \mathcal{R} \Rightarrow \bar{\Pi}_i^s \text{ by } B_i^s (r) = \{\bar{\pi}_i^s \in \bar{\Pi}_i^s | \bar{\pi}_i^s \text{ maximizes } f_i^s (\cdot, r)\}.
$$

Since $f_i^s$ is continuous in both arguments, $B_i^s$ is non-empty, compact-valued and upper hemicontinuous by the Theorem of Maximum. Obviously, $B_i^s$ is also convex-valued for all $r \in \mathcal{R}$ since $V_i^s (r) = V_i^s (\cdot) (r)$ for all $L_{i\phi}$ and $L_i^s$ in the support of any $\bar{\pi}_i^s \in B_i^s (r)$. Let $B^s = \times_i B_i^s$.

Denote the combinations of all players’ coalition formation strategy by $\pi^s \in \times_i \bar{\Pi}_i^s = \bar{\Pi}^s$. For each $\pi^s \in \bar{\Pi}^s$, define a function $\Psi_i^s (\cdot ; \pi^s) : \mathcal{R} \to \mathcal{R}$ by

$$\Psi_i^s (r; \pi^s) = \frac{s_i + (T - 1) \sum_{j \in I} s_j \sum_{\phi = 1}^{L_i^s} \pi_j^s (L_{i\phi}) V_i^s (\pi^s (r))}{T}.
$$

Since $\Psi_i^s (\cdot ; \pi^s)$ is a contraction mapping from a compact set to itself, it has a unique fixed point. Let $\psi^s : \bar{\Pi}^s \to \mathcal{R}$ be a fixed-point operator of the function $\Psi_i^s$ such that $\psi^s (\bar{\pi}^s) = r$ if and only if $\Psi_i^s (r; \bar{\pi}^s) = r$. Notice that $\psi^s$ is non-empty and continuous in $\bar{\pi}^s$ by continuity of $\Psi_i^s$ in $\bar{\pi}^s$. Define a correspondence $B^s \circ \psi^s : \bar{\Pi}^s \Rightarrow \bar{\Pi}^s$ by $B^s \circ \psi^s (\bar{\pi}^s) = \{\pi^s \in \bar{\Pi}^s | \pi^s \in B^s (\psi^s (\bar{\pi}^s))\}$. Since $\bar{\Pi}^s$ is a compact convex set $B^s \circ \psi^s$ is upper hemicontinuous with non-empty compact convex-valued, $B^s \circ \psi^s$ has a fixed point by Kakutani’s fixed point theorem. Denote any fixed point of $B^s \circ \psi^s$ by $\pi^{ss}$ and $\psi (\pi^{ss}) = r^{ss}$. For each $i$ and $i$’s decisive coalition $L_{i\phi}$, let $V_i^s (r^{ss}) = V_i^s (\pi^{ss})$.

For each $i \in I$, construct $\pi_i^s (V) = \pi_i^s (L_{i\phi})$ for $V = V_i^s (\pi^{ss})$ and $\pi_i^s (V) = 0$ otherwise. Also let $a_i^s (V) = \{\text{Accept}\}$ if and only if $V_j \geq r_j^s$. Then we have the following Lemma.

**Lemma 8.** For $s \in S_N^c$, $(\pi^{ss}, a^{ss})$ is an SSPE of $G(s)$ and $r^{ss} = r^s (V)$.

**Proof of Lemma 8.** For any $i$ and proposal $V$ such that $\pi_i^s (V) > 0$, we know that $V = V_i^s (\pi^{ss})$ and there is a decisive coalition $L_{i\phi}^s \in L_i^s$, where $V_j = V_j^s (\pi^{ss}) \geq r_j^s$ for all $j \neq i \in L_{i\phi}^s$ by construction of $V_i^s$. Given
r^{s^*}. Thus any i’s proposal V in the support of \( \pi_i^s \) is approved by a decisive coalition in undominated voting strategies.

Since \( \bar{\pi}_i^{s^*} \) maximizes \( f_i^{\pi_i^s} \) and \( f_i^{s^*} \) is linear in \( V_i^{s_i,ib}(r^{s^*}) \) associated with \( L_{ib}^s \) selected with a positive probability, we know that \( V_i^{s_i,ib}(r^{s^*}) = V_i^{s_i,eb}(s^*) \) for any \( L_{ib}^s \) and \( L_{ib}^i \) in the support of \( \bar{\pi}_i^{s^*} \). Accordingly, \( V_i = V'_i \) for all V and \( V' \) in the support of \( \pi_i^{s^*} \), and we only need to show that a single V such that \( \pi_i^{s^*}(V) > 0 \) maximizes \( V_i \) among all possible proposals that are acceptable by any decisive coalition. Let \( V_i^{s_i,eb} \) be a proposal associated with a decisive coalition \( L_{ib}^s \) such that \( \pi_i^{s^*}(V_i^{s^*}) > 0 \). By construction of \( V_i^{s_i,eb} \), we have either \( V_i^{s_i,eb} = \frac{1}{2} \) or \( V_i^{s_i,eb} < \frac{1}{2} \). If \( V_i^{s_i,eb} = \frac{1}{2} \), \( \pi_i^{s^*} \) is obviously optimal since \( V_i \leq \frac{1}{2} \) for all V by the definition of \( \mathcal{V} \). If \( V_i^{s_i,eb} < \frac{1}{2} \), \( \pi_i^{s^*} \) is again optimal since \( V_i^{s_i,eb} = r^{s^*} \) for all \( j \in L_{ib}^s \) and \( L_{ib}^s \) is i’s decisive coalition with minimum demands \( \sum_j e_{L_{ib}^s}\{j\} r^{s^*}_{j} \). Any proposal \( V' \) such that \( V'_i > V_i^{s_i,eb} \) cannot be accepted by any decisive coalition.

Finally, it is easy to see that \( r^{s^*} \) is a \( T \)-period reservation value vector in \( (\pi^{s^*}, a^{s^*}) \) since \( r^{s^*} \) is the unique fixed point of \( \Psi_i^{s,T} \) given \( \pi^{s^*} \). Writing \( \Psi_i^{s,T} \), replacing \( \bar{\pi}^{s^*} \) with \( \pi^{s^*} \), we have

\[
\Psi_i^{s,T}(r^{s^*}, \pi^{s^*}) = \frac{s_i + (T - 1) \sum_{j \in L} s_j \sum_{\pi \in \mathcal{V} \pi_i^{s^*}(\pi) > 0} \pi_j^{s^*}(\pi) V_i}{T} = r^{s^*},
\]

for each i. Thus the strategy profile \( (\pi^{s^*}, a^{s^*}) \) derived from \( \bar{\pi}^{s^*} \) is an SSPE of \( G(s) \), and \( r^{s^*} \) is an SSPE \( T \)-period reservation value vector of \( G(s) \) for each \( s \in S_{NC}^e \).

Before moving on to \( S_{NC}^e \), I prove the following lemma for \( S_{NC}^e \):

**Lemma 9.** Let \( s \in S_{NC}^e \). Consider \( i \in I \) with \( s_i > 0 \) and \( V \in \mathcal{V} \) such that \( \pi_i^{s^*}(\pi) > 0 \) and \( V_i < \frac{1}{2} \). Let \( M = \{ j \in I \mid s_j > 0 \} \) and \( L \subseteq M \) be the decisive coalition containing \( i \) associated with \( V \). Then the following statements hold true:

(i) \( \sum_{k \in L} s_k - s_j < \frac{1}{2} \) for all \( j \in L \setminus \{i\} \);
(ii) \( V_i + \sum_{j \in L} V_j > \frac{1}{2} \) for all \( j \in L \setminus \{i\} \).

**Proof of Lemma 9.** Let \( \tilde{L} = L \setminus \{i\} \) and \( L = [M \setminus \tilde{L} \setminus \{i\}] \).

(i) By \( \pi^{s^*} \) being an SSPE of \( G(s) \), \( \pi_i^{s^*}(\pi) > 0 \) and \( V_i < \frac{1}{2} \), we know that \( \sum_{j \in \tilde{L}} V_j > \frac{1}{2} \) and \( V_j = r^{s^*}_j > 0 \) for all \( j \in \tilde{L} \). Suppose on the contrary to the statement (i) that \( \sum_{k \in L} s_k - s_j > \frac{1}{2} \) for some \( j \in \tilde{L} \). By \( s_j > 0 \), then, \( \sum_{k \in L} s_k - s_j > \frac{1}{2} \) and \( L \setminus \{j\} \) is a decisive coalition. It follows that there is a proposal \( V' \) such that \( V'_j = r^{s^*}_j \) for all \( j \in \tilde{L} \setminus \{j\} \). \( V'_j = 0 \) and \( V'_j = \min\{\frac{1}{2}, 1 - \sum_{k \in L} r^{s^*}_k\} > 1 - \sum_{k \in L} r^{s*}_k = V_i \), which leads to a contradiction that \( \pi^{s^*} \) is not optimal.

(ii) By way of contradiction, suppose that \( V_i + \sum_{j \in \tilde{L}} V_j < \frac{1}{2} \) for some \( j \in \tilde{L} \). Then, \( \sum_{k \in \tilde{L} \setminus \{j\}} V_k = \sum_{k \in \tilde{L} \setminus \{j\}} r^{s^*}_k > \frac{1}{2} \) and \( \sum_{k \in L} r^{s^*}_k + r^{s^*}_j < \frac{1}{2} \) by \( V_k = r^{s^*}_k \) for \( k \in \tilde{L} \) and \( r^{s^*}_j > 0 \). By (i), \( s_j + \sum_{k \in \tilde{L}} s_k > \frac{1}{2} \) and \( L' = \{i, j\} \cup L \) is a decisive coalition. Then there is a proposal \( V' \) that will be accepted by all members in \( L' \) such that \( V'_i = \frac{1}{2} > V_i \), \( V'_j = r^{s^*}_j \) for \( i \in L' \setminus \{i\} \) and \( V'_j = 0 \) for \( k \in L' \), which leads to a contradiction that \( \pi_i^{s^*} \) is not optimal.

**SSPE of G(s) in s ∈ S_{NC}^e**  Notice that in every \( s \in S_{NC}^e \), there is a single player i who can form a decisive coalition with any other player j who has a positive voting weight. That is, \( \exists i \in I \) such that \( s_i + s_j > \frac{1}{2} \) for all \( j \neq i \in \tilde{J} \), where \( \tilde{J} = \{ i \in I \mid s_j > 0 \} \). To see this, suppose that there are \( |\tilde{J}| < 4 \) players having a positive status quo share (voting weight).\(^8\) Without loss of generality, suppose \( s_1 + s_2 > \frac{1}{2} \). Then \( s_j + s_k < \frac{1}{2} \) for any two players \( j, k \neq 1, 2 \in \tilde{J} \). By \( s \in S_{NC}^e \), either \( s_1 + s_j < \frac{1}{2} \) or \( s_2 + s_j > \frac{1}{2} \). Consider the case \( s_1 + s_j > \frac{1}{2} \).

\(^8\)The case with 3 players with positive status quo shares is obvious, since \{1, 2\}, \{1, 3\}, \{2, 3\} are all decisive in any noncollegial states.
Then \(s_2 + s_k < \frac{1}{2}\) for all \(k \neq 1, 2, j \in \bar{J}\). It follows that \(s_1 + s_2 > \frac{1}{2}, s_1 + s_j > \frac{1}{2}\), and \(s_1 + s_k > \frac{1}{2}\) for all \(k \neq 1, 2, j \in \bar{J}\). If we have chosen \(s_2 + s_j > \frac{1}{2}\), the same is true for player 2.

For \(s \in S_{NC}^e\), I specify an SSPE proposal strategy profile \(\bar{\pi}^s\) and an SSPE reservation value vector \(r^s\) directly. Let the player who can form a decisive coalition of size two with every other player be player \(i\), and let \(J = \bar{J}\{i\}\). Note that \([i, j]\), for all \(j \in J\), and the set \(J\) are decisive coalitions. For \(j \in J\), denote \([i, j]\) by \(L_{ij}\).

Consider the following the coalition formation strategy \(\bar{\pi}^s\). For all \(j \in J\),

\[
\bar{\pi}^s_j(L_{ij}) = \frac{s_j}{s_i} \left[ 1 - \left( \frac{1 - 2s_i}{1 - s_i} \right) \sum_{h \in J \backslash j} \frac{s_h}{1 - s_i - s_h} \right]
\]

\[
\bar{\pi}^s_j(L_{ij}) = \frac{s_i}{1 - s_i} \text{ and } \bar{\pi}^s_j(J) = \frac{1 - 2s_i}{1 - s_i}.
\]

In words, \(\bar{\pi}^s_j(L_{ij})\) is the probability that player \(i\) chooses each \(j \in J\) as a coalition partner. One can easily verify that \(\sum_{j \in J} \pi^s_i(L_{ij}) = 1\) by \(\sum_{j \in J} s_j = 1 - s_i\). Similarly, \(\bar{\pi}^s_j(L_{ij})\) is the probability that \(j \in J\) chooses player \(i\) as a coalition partner, and \(\bar{\pi}^s_j(J)\) is the probability that \(j\) forms the decisive coalition \(J\). Again, \(\bar{\pi}^s_j(L_{ij}) + \bar{\pi}^s_j(J) = 1\).

For \(i \neq j \in J\), let \(V^s_{ij}\) be the proposal associated with coalition \(L_{ij}\), where \(V^s_{ij} = V_{ij} = \frac{1}{2}\). For every \(j \in J\), let \(V^s_{ij}\) be \(j\)'s proposal associated with coalition \(J\), where

\[
V^s_{ij} = 0, \quad V^s_{ij} = \frac{1}{2} \quad \text{and} \quad V^s_{ij} = s_h + \frac{s_h}{1 - s_i - s_j}(s_i + s_j - \frac{1}{2}) \text{ for } h \notin j \in J.
\]

Construct \(\pi^s\) in the following way: For player \(i\), \(\pi^s_i(V) = \bar{\pi}^s_i(L_{ij})\) if \(V = V^s_{ij}\) for \(j \in J\) and \(\pi^s_i(V) = 0\) otherwise. For each \(j \in J\), \(\pi^s_j(V) = \bar{\pi}^s_j(L_{ij})\) if \(V = V^s_{ij}\); \(\pi^s_j(V) = \bar{\pi}^s_j(J)\) if \(V = V^s_{ij}\); and \(\pi^s_j(V) = 0\) otherwise. The following lemma establishes that \(\pi^s\) is an SSPE proposal strategies in \(s \in S_{NC}^e\).

**Lemma 10.** In \(s \in S_{NC}^e\), \(\pi^s\) is an SSPE proposal strategy profile in undominated voting strategies \(a^s\). Furthermore, the \(T\)-period reservation value vector \(r^{s, T^*} = s\) in the SSPE.

**Proof of Lemma 10.** First, notice that for any proposer \(j \in \bar{J}\) (including \(i\)), any proposal \(V\) that is used with a positive probability assigns \(V_j = \frac{1}{2}\) to the proposer. By the definition of \(V\), all such \(V\) is optimal.

Next, assume that every proposal \(V\) used with a positive probability is approved by at least one decisive coalition (which will be verified at the end of the proof) and compute the reservations values in the given proposal strategy \(\pi^s\) for \(i \neq j \in J\).

\[
r^s_i(T) = \frac{s_i + (T - 1) \sum_{k \neq i} s_k \sum_{j \in J} \pi^s_i(V_j) V_j}{T} = \frac{s_i}{T} + \frac{T - 1}{T} \left[ s_i \sum_{j \in J} \bar{\pi}^s_i(L_{ij}) V^s_{ij} + \sum_{j \in J} s_j \bar{\pi}^s_i(L_{ij}) V^s_{ij} \right] = \frac{s_i}{T} + \frac{T - 1}{T} \left[ s_i + \frac{s_i}{2} \right] = s_i
\]

\[
r^s_j(T) = \frac{s_j}{T} + \frac{T - 1}{T} \left[ s_j \bar{\pi}^s_i(L_{ij}) V^s_{ij} + \sum_{h \in J \backslash j} s_h \bar{\pi}^s_i(J) V^s_{ij} \right] = \frac{s_j}{T} + \frac{T - 1}{T} \left[ s_j \left( \frac{s_j}{2} + \frac{s_j}{2} \left( 1 - \frac{1 - 2s_i}{1 - s_i} \sum_{h \in J \backslash j} \frac{s_h}{1 - s_i - s_h} \right) \right) + \sum_{h \in J \backslash j} s_h s_i \left( \frac{1 - 2s_i}{1 - s_i} \right) \left[ \frac{s_i + s_h - \frac{1}{2}}{1 - s_i - s_h} \right] \right] = s_j.
\]
We have \( r^{i,T*} = s \) as desired.

Finally, we need to show that all proposals are accepted by a decisive coalition in undominated voting strategies given the reservation values. For the proposals associated with \( L_{ij} \) for any \( j \in J \), such proposals are accepted by \( i \) and \( j \) since they receive \( \frac{1}{2} \) and \( s_i, s_j < \frac{1}{2} \) in any noncollegial state. The proposals associated with \( J \) are accepted by \( J \) since any proposer \( j \in J \) receives \( \frac{1}{2} > s_j \) and any other player \( h \neq j \in J \) receives \( s_h = r^{h,T*} \). □

Now we prove the existence of equilibrium of \( \Gamma \). Note that \( V \subset X \) by the definition \( V = \{ V \in \mathbb{R}^n | \sum_{i \in I} V_i = 1, 0 \leq V_i \leq \frac{1}{2}, \forall i \in I \} \).

**Lemma 11.** Construct \( \mu^* \) in the following way: For tyrannical, dictatorial and collegial states \( s \), let \( \mu^*(s) \) be as described in Lemmas 1 and 2. For noncollegial states \( s \), let \( \mu^*_i(y|s) = \pi^*_i(y) \) for \( y \in V \) and \( \mu^*_i(y) = 0 \) for \( y \in X \setminus V \) for all \( i \), where \( \pi^*_i \) is an SSPE proposal strategy profile of \( G(s) \). Then,

(i) \( y \in A^*(s) \) for all \( y \in X \) such that \( \mu^*_i(y|s) > 0 \);

(ii) \( \{y_i^*, A_i^*\}_{i \in I} \) is an equilibrium.

**Proof of Lemma 11.** Lemma 1 and 2 show that the proposals in tyrannical, dictatorial, oligarchic and collegial states are optimal and accepted by a decisive coalition. Also note that the players in collegial and noncollegial states never accept other players’ dictatorial or tyrannical divisions. Thus, we focus on noncollegial states with the constraint that no proposer in noncollegial states can assign more than a half to anyone including the proposer oneself.

(i) Given \( \mu^* \) and assuming that all proposals that are proposed with a positive probability are accepted by a decisive coalition, we first compute the \( T \)-period expected utility \( U^{rT}(s) \) from \( s \in S'_{NC} \). Recall that \( \pi^*_i(y) > 0 \) for any \( i \in I \) is either oligarchic or collegial because \( y_i = \frac{1}{2} \). From (7) and (11), we know that \( U^{rT}_j(y) = U^{rT-1}_j(y) = y_j \) for all \( j \) in any oligarchic or collegial state \( y \). Since \( \mu^*_i(y|s) = \pi^*_i(y) \) for all \( i \),

\[
U^{rT}_i(s) = \frac{s_i + (T - 1)\mu^{rT-1}(s)}{T} = \frac{s_i}{T} + \frac{T - 1}{T} \sum_{j \in I} s_j \sum_{y: \mu^*_j(y) > 0} \mu^*_j(y|s)U^{rT-1}_i(y)
\]

\[
= \frac{s_i}{T} + \frac{T - 1}{T} \sum_{j \in I} s_j \sum_{y: \pi^*_i(y) > 0} \pi^*_i(y)y_j = r^{i,r}_i = s_i,
\]

(19)

where the last equality comes from Lemma 10. Now see that every proposal is accepted by at least one decisive coalition. Since for any proposal \( y \) such that \( \pi^*_i(y) > 0 \), hence \( \mu^*_i(y|s) > 0 \), \( y \) is either oligarchic or collegial in \( s \in S'_{NC} \). If \( y \) is oligarchic, it is accepted by the two oligarchs. If \( y \) is collegial, there is a decisive coalition \( J \) containing \( i \) such that, for all \( j \in J \),

\[
y_j = U^{rT}_j(y) > U^{rT}_j(s) = s_j \iff \lim_{T \to \infty} T[U^{rT}_j(y) - U^{rT}_j(s)] > 0 \iff y > j, s,
\]

by the construction of SSPE proposal \( y \) in \( s \in S'_{NC} \), where \( U^{rT}_j(y) = y_j \) comes from (11) and \( U^{rT}_j(s) = s_j \) from (19).

In \( s \in S'_{NC} \), any \( y \) such that \( \pi^*_i(y) > 0 \) for any \( i \in I \) is either oligarchic, collegial or \( y \in X'_{NC} = S'_{NC} \) by Lemma 9. Again, \( U^{rT}_i(y) = U^{rT-1}_i(y) = y_i \) if \( y \) is an oligarchic or collegial division, and the same holds for
\[ y \in S_{NC}^c \text{ by (19). Thus,} \]
\[ U_i^{sT}(s) = \frac{s_i + (T - 1)\mu_i^{sT-1}(s)}{T} = \frac{s_i}{T} + \frac{T - 1}{T} \sum_{j \in I} s_j \sum_{y : \pi_j^*(y) > 0} \sum_{y : \pi_j^*(y) > 0} \frac{\mu_j^*(y|s) U_i^{sT-1}(y)}{T} \]
\[ = \frac{s_i}{T} + \frac{T - 1}{T} \sum_{j \in I} s_j \sum_{y : \pi_j^*(y) > 0} \pi_j^*(y) y_i = r_i^{sT^*}. \] (20)

By construction of proposal \( y \) such that \( \pi_i^*(y) > 0 \) in \( s \in S_{NC}^{ne} \), there is a decisive coalition \( J \) containing \( i \) such that \( y_j \geq r_j^{sT^*} \) for all \( j \in J \). Thus,
\[ y_j = U_j^{sT}(y) \geq U_j^{sT}(s) = r_j^{sT^*} \iff \lim_{T \to \infty} T[U_j^{sT}(y) - U_j^{sT}(s)] \geq 0 \iff y \geq_j s, \]
and every player’s proposals in noncollegial states are accepted in stage-undominated voting strategies.

(ii) We need to show that any \( y \) such that \( \mu_i^*(y|s) > 0 \) gives any proposer \( i \) the highest expected utility \( U_i^*(y) \) among all socially acceptable proposals. For the optimality of proposals in tyrannical, dictatorial, oligarchic, and collegial states, see Lemmas 1 and 2. Thus, focus on noncollegial states. In \( s \in S_{NC}^c \), every player’s proposal is optimal since \( y_i = \frac{1}{2} \) for all \( y \) with \( \pi_i^*(y) = \mu_i^*(y|s) > 0 \). In \( s \in S_{NC}^{ne} \), if \( y \) is oligarchic or collegial, \( y \) is optimal for the same reason. So consider the case that \( y_i < \frac{1}{2} \) and \( \mu_i^*(y|s) > 0 \) for some \( i \). Then \( y \in S_{NC}^c \) by Lemma 9, and \( U_i^{sT}(y) = y_j \) for all \( j \neq i \) such that \( y_j > 0 \) by (19). Also, \( y_j = r_j^{sT^*} \) if \( y_i < \frac{1}{2} \) by the construction of SSPE proposals in \( s \in S_{NC}^{ne} \). Then \( y \) is an optimal proposal of \( i \): It gives \( U_i^{sT}(s) = r_i^{sT^*} \) to a set of decisive coalition partners \( J \setminus \{i\} \) with the smallest sum of expected utilities \( \sum_{J \setminus \{i\}} r_j^{sT^*} = \sum_{J \setminus \{i\}} U_j^*(y) \) and gives \( i \) the remainder \( U_i^*(y) = 1 - \sum_{J \setminus \{i\}} U_j^*(y) \). There is no decisive coalition for \( i \) to choose with lower demands since \( \pi_i^{sT^*} \) is an SSPE of \( G(s) \), \( \pi_i^{sT}(y) > 0 \) and \( y \) is associated with the coalition with the lowest demands. ■