1. Assume that \( Z \sim \text{N}(0, I_n) \). Let \( W = \Sigma_{i=1}^{n} Z_i \), \( X = \Sigma_{i=1}^{n} Z_i^2 \), and \( Y = \Sigma_{i=1}^{n} (Z_i - \overline{Z})^2 \).
   a. Are \( W \) and \( X \) statistically independent?
   b. Are \( W \) and \( Y \) statistically independent?
   c. Are \( X \) and \( Y \) statistically independent?

2. Assume that we have a random sample from a Poisson distribution. The Poisson density function is:
   \[
   f(X) = \frac{\alpha^X e^{-\alpha}}{X!} \quad \text{for } X = 0, 1, 2, \ldots, \infty
   \]
   The unconstrained ML estimator of \( \alpha \) is \( \overline{X} \). Under the restriction \( \alpha = \alpha_0 \), the constrained ML estimator of \( \alpha \) is just \( \alpha_0 \).
   a. Define the LR critical region (for the general case).
   b. Express the LR critical region for testing the hypothesis pair \( H_0: \alpha = \alpha_0 \) versus \( H_A: \alpha \neq \alpha_0 \) in terms of an inequality involving \( \overline{X} \).
   c. Is this a one-sided or two-sided critical region? Explain.

3. Consider the constrained least squares estimator,
   \[
   \hat{\beta} = \hat{\beta} - (X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}(R\hat{\beta} - r)
   \]
   where the \( m \) linearly independent constraints, \( R\beta = r \), may or may not be valid.
   a. Given that OLS is unbiased, find \( E(\tilde{\beta}) \). When is \( \tilde{\beta} \) an unbiased estimator of \( \beta \)? When is \( \tilde{\beta} \) a biased estimator of \( \beta \)?
   b. Show that the covariance matrix of \( \tilde{\beta} \) is
   \[
   \sigma^2[(X'X)^{-1} - (X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}R(X'X)^{-1}].
   \]
   c. Show that the difference between the covariance matrix of \( \hat{\beta} \) and the covariance matrix of \( \tilde{\beta} \) is a positive semi-definite matrix. What does this mean?
   d. Evaluate the following statement: “One should never use the constrained least squares estimator since imposition of the constrain might lead to biased estimates.”
4. Assume that we have a pair of samples \((Y_1, X_1)\) and \((Y_2, X_2)\) as per the discussion of the Chow test. If we wish to estimate separate models for each sample, we can stack them as

\[ Y = X\beta + \epsilon \]

where

\[
Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}, \quad X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \quad \epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}
\]

Note that if we expand the matrix products we get.

\[
\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X_1\beta_1 + \epsilon_1 \\ X_2\beta_2 + \epsilon_2 \end{bmatrix}
\]

a. Show that the linear restrictions \(\beta_1 = \beta_2\) may be written in the form \(R\beta = r\).

b. Show that \(\hat{\beta} = (X'X)^{-1}X'Y\) reduces to

\[
\hat{\beta} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} (X_1'X_1)^{-1}X_1'Y_1 \\ (X_2'X_2)^{-1}X_2'Y_2 \end{bmatrix}
\]

c. Show that \(e'e = e_1'e_1 + e_2'e_2\).

5. Assume that we have a random sample from a Poisson distribution. The Poisson density function is:

\[ f(X) = \frac{\alpha^Xe^{-\alpha}}{X!} \quad \text{for } X = 0, 1, 2, \ldots, \infty \]

a. Find the NP critical region for testing \(H_0: \alpha = \alpha_0\) versus \(H_A: \alpha > \alpha_0\).

b. Determine whether the NP critical region in part (a) is UMP for testing \(H_0: \alpha \leq \alpha_0\) versus \(H_A: \alpha > \alpha_0\).
Theorems

1. If $A$ is a $q \times q$ non-singular symmetric matrix and $B$ is a $q \times r$ matrix, then:
   a. $B'B$ is positive definite if $B$ has rank $r$.
   b. $A^{-1}$ is positive definite if $A$ is positive definite.
   c. the rank of $AB$ equals the rank of $B$.
   d. $B'AB$ is positive definite if $B$ has rank $r$.
   e. $B'AB$ is positive semi-definite if $B$ has rank $q$. 