References


Change-in-Variable Method

We have a random variable \( X \) with known density function \( f(X) \) and sample space \( \Sigma_X = \{ X | f(X) > 0 \} \). A new variable \( Y \) is defined as a monotonic transformation of \( X \). That is, \( Y = g(X) \), where either \( g'(X) > 0 \) for all \( X \) or \( g'(X) < 0 \) for all \( X \). These conditions are sufficient to insure the existence of an inverse transformation, \( X = G(Y) \). (Note that the inverse transformation is often denoted \( X = g^{-1}(Y) \). In general, this will not be \( g(X) \) to the power minus one.) The sample space of \( Y \) is \( \Sigma_Y = \{ Y | Y = g(X) \text{ for } X \text{ in } \Sigma_X \} \). We wish to find the density function of the new random variable \( Y \).

Discrete Case

If \( x \) is an arbitrary element of \( \Sigma_X \), and \( y = g(x) \) the corresponding element of \( \Sigma_Y \), then

\[
f_Y(y) = P(Y = y) = P(X = G(y)) = f_X[G(y)]
\]

for \( y \) in \( \Sigma_Y \) and hence \( x = G(y) \) in \( \Sigma_X \).

Continuous Case

In the continuous case, probability is given by integrals of the density function. Hence, we will examine the corresponding distribution functions. Let \( x \) denote an arbitrary value for
the argument of the distribution function of $X$, and let $Y = g(X)$. Then,

(a) if $g'(X)>0$ for all $X$, we have

$$P(Y \leq y) = P(X \leq x)$$

or in terms of the required integrals

$$\int_{-\infty}^{y} f_Y(\eta) d\eta = \int_{-\infty}^{x} f_X(\lambda) d\lambda$$

since $X = G(Y)$, integration by substitution gives

$$\int_{-\infty}^{y} f_Y(\eta) d\eta = \int_{y}^{\infty} f_X[G(\eta)]G'(\eta) d\eta$$

differentiation with respect to $Y$ gives

$$f_Y(y) = f_X[G(y)]G'(y) \quad \text{for } y \in \Sigma_Y$$

$$= f_X[G(y)] |G'(y)| \quad \text{for } y \in \Sigma_Y$$

(b) if $g'(X)<0$ for all $X$, we have

$$P(Y \leq y) = P(X \geq x)$$

or in terms of the required integrals

$$\int_{-\infty}^{y} f_Y(\eta) d\eta = \int_{x}^{\infty} f_X(\lambda) d\lambda$$

since $X = G(Y)$, integration by substitution gives

$$\int_{-\infty}^{y} f_Y(\eta) d\eta = \int_{-\infty}^{y} f_X[G(\eta)]G'(\eta) d\eta$$

differentiation with respect to $Y$ gives

$$f_Y(y) = -f_X[G(y)]G'(y) \quad \text{for } y \in \Sigma_Y$$

$$= f_X[G(y)] |G'(y)| \quad \text{for } y \in \Sigma_Y$$

In summary, the density function for the monotonic transformation $Y$ may be expressed in terms of the density for $X$ as

$$f_Y(y) = f_X[G(y)] |G'(y)| \quad \text{for } y \in \Sigma_Y$$

Discrete example

Assume that $X$ is Bernoulli with parameter $P$. The density function of $X$ is

$$f(x) = P^x (1-P)^{1-x}$$
for $0 \leq P \leq 1$ and $X \in \{0,1\}$. Let $Y = X + 1$. The sample space of $Y$ is $\{1,2\}$. The inverse transformation is $X = Y - 1$. The density function of $Y$ is

$$f(Y) = P(Y-1)(1 - P)^{(Y+1)} = P(Y-1)(1 - P)^{(2 - Y)}$$

for $0 \leq P \leq 1$ and $Y \in \{1,2\}$.

**Continuous example**

Assume that $X \sim U(0,1)$. The density function of $X$ is

$$f(X) = I(X)$$

where the variable $I(X) = 1$ if $X \in (0,1)$ and $I(X) = 0$ if $X \notin (0,1)$. Let $Y = \alpha X$ for $\alpha > 0$. The sample space of $Y$ is the open interval $(0, \alpha)$. The inverse transformation is $X = Y/\alpha$. The Jacobian of the transformation is $1/\alpha$. The density function of $Y$ is

$$f(Y) = \frac{1}{\alpha} I\left(\frac{Y}{\alpha}\right) = \frac{1}{\alpha} J(Y)$$

where $J(Y) = 1$ if $Y \in (0, \alpha)$ and $J(Y) = 0$ if $J \notin (0, \alpha)$.

**Continuous example**

Assume that $X$ is exponential with parameter $\lambda$. The density function of $X$ is

$$f(X) = \lambda \exp(-\lambda X)$$

for $\lambda > 0$ and $X \in \mathbb{R}^+$. Let $Y = X^2$. The sample space of $Y$ is also $\mathbb{R}^+$. This transformation is monotone over $\mathbb{R}^+$. The inverse transformation is $X = \sqrt{Y}$ and the Jacobian of the transformation is $(2\sqrt{Y})^{-1}$. The density function of $Y$ is

$$f(Y) = \frac{\lambda}{2\sqrt{Y}} \exp(-\lambda \sqrt{Y})$$

for $\lambda > 0$ and $Y \in \mathbb{R}^+$. 
Multivariate Change-in-Variable Method

We have a random vector \( X \) with known joint density function \( f_X(X) \) and sample space \( \Sigma_X = \{ X \mid f_X(X) > 0 \} \). A new vector \( Y \) is defined as a set of monotonic transformations of \( X \). That is, \( Y = g(X) \), where we assume the existence of a set of inverse transformation, \( X = G(Y) \). In general, each element in \( Y \) will depend on the entire vector \( X \), and vice versa. The Jacobian of the inverse transformations is \( J = \frac{\partial X}{\partial Y} \), which is generally a function of \( Y \). The sample space of \( Y \) is \( \Sigma_Y = \{ Y \mid Y = g(X) \text{ for } X \in \Sigma_X \} \).

We wish to find the density function of the random vector \( Y \). If \( X \) denotes an arbitrary vector in \( \Sigma_X \), and \( Y = g(X) \) the corresponding vector in \( \Sigma_Y \), then in the discrete case, the joint density of \( Y \) is:

\[
f_Y(y) = f_X[G(y)] \text{ for } y \in \Sigma_Y \text{ and hence } X = G(y) \text{ in } \Sigma_X,
\]

and in the continuous case, the joint density of \( Y \) is:

\[
f_Y(y) = f_X[G(y)] \lvert J \rvert \text{ for } y \in \Sigma_Y \text{ and hence } X = G(y) \text{ in } \Sigma_X.
\]

Linear Transformations

Given a random n-vector \( X \) with mean vector \( \mu_X \) and covariance matrix \( \Sigma_X \), consider the set of \( n \) linear transformations \( Y = AX \). The random vector \( Y \) has mean vector \( \mu_Y = A \mu_X \) and covariance matrix \( \Sigma_Y = A \Sigma_X A' \). If \( A \) is non-singular, then there exist a set of inverse transformations \( X = A^{-1}Y \). The Jacobian matrix is just \( A^{-1} \), and given the joint density \( f_X(X) \), the joint density of \( Y \) is just \( f_Y(Y) = f_X(A^{-1}Y) \lvert A^{-1} \rvert \).