Section 1 - Definitions
ECO6936: Topics In Microeconomic Theory

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Definition 1 *A game is a formal representation of a situation in which 2 or more players interact in a setting of strategic interdependence. Or, the payoff of at least one of the players depends on the actions of another player.*

Choice Axioms.

1. Completeness: for all $x, y \in \mathcal{L}$, we have that $x \gtrsim y$ or $y \gtrsim x$ or both.
2. Transitivity: For all $x, y, z \in \mathcal{L}$, if $x \gtrsim y$ and $y \gtrsim z$ then $x \gtrsim z$.
3. Continuity: The preference relation $\gtrsim$ is continuous on the space of simple lotteries, $\mathcal{L}$ is continuous if for any $L, L', L'' \in \mathcal{L}$, the sets:

$$\{\alpha \in [0,1] : \alpha L + (1-\alpha)L' \gtrsim L''\} \subset [0,1]$$
$$\{\alpha \in [0,1] : L'' \gtrsim \alpha L + (1-\alpha)L'\} \subset [0,1]$$

are closed. Alternate definition. Order all outcomes in the set as $b_1 \gtrsim b_2 \gtrsim ... \gtrsim b_n$. Then take any other gamble g. It must be the case that if we take the compound lottery with $\alpha \in [0,1]$, $\alpha b_1 + (1-\alpha)b_2$, if $\alpha = 1$ then it should be preferred to g, if $\alpha = 0$ then g must be preferred to it. If continuity holds, there must be some α such that $\alpha b_1 + (1-\alpha)b_2\tilde{}g$. Note this could be at $\alpha = 1$ or 0.

4. Independence (sometimes substitution): The preference relation $\gtrsim$ on the space of simple lotteries $\mathcal{L}$ satisfies the independence axiom if for all $L, L', L'' \in \mathcal{L}$ and $\alpha \in (0,1)$, we have

$$L \gtrsim L' \text{ if and only if } \alpha L + (1-\alpha)L'' \gtrsim \alpha L' + (1-\alpha)L''$$

Definition 2 *Extensive form game. An EFG game tree consists of the following characteristics.*

1. A set of nodes, X, a set of possible actions A, and a set of players $\{1, ..., I\}$.

2. A function $p : X \to \{X \cup \emptyset\}$ specifying a simple immediate predecessor of each node. $p(x)$ is nonempty for all $x \in X$ except one, x_0 (root node and it is unique). $s(x) = p^{-1}(x)$ defines the successor nodes. $p(x)$ and $s(x)$ are set of all predecessor and successor nodes of a node x and they must be disjoint. Set of terminal nodes are $T = \{x \in X : s(x) = \emptyset\}$. All other nodes $X \backslash T$ are decision nodes.

3. A function $\alpha : X \backslash \{x_0\} \to A$. Giving the action that leads to any non-initial node. $c(x) = \{a \in A : a = \alpha(x') \text{ for some } x' \in s(x)\}$. So $c()$ is a set of actions available at each node.

4. A collection of information sets H and a function $h : X \to H$ assigning each node into an information set. Note all nodes in the same information set must have the same choices.

5. A function $\iota : H \to \{0, ...I\}$ assigning each information set in H to the player who moves at the decision nodes contained in H.

6. A function $\rho : H_0 \times A \to [0, 1]$ assigning probabilities to actions at information sets where nature moves. Nature's information set.

7. A collection of payoff functions $u = \{u_1(.), ..., u_I(.)\}$ assigning utilities to the players at each terminal node.

Formally the game is defined by $\Gamma_E = \{X, A, I, p(.), \alpha(.), H, h(.), \iota(.), \rho(.), u\}$.

Definition 3 *Let H_i denote the collection of player i's information sets, A the set of possible actions in the game and $C(h) \subset A$ the set of actions at information set h. A strategy for player i is a function $s_i : H_i \to A$ such that $s_i(h) \in C(h)$ for all $h \in H_i$*

Definition 4 *Let H_i denote the collection of player i's information sets, A the set of possible actions in the game and $C(h) \subset A$ the set of actions at information set h. A strategy for player i is a function $s_i : H_i \to A$ such that $s_i(h) \in C(h)$ for all $h \in H_i$*

Definition 5 *For a game with I players, the normal form representation Γ_N specifies for each players i a set of strategies S_i and a payoff function $u_i(s_1, ..., s_I)$ giving the VNM utility levels associated with the outcome arising from strategies $(s_1, ..., s_I)$ (i.e. the strategies played by all players). Formally $\Gamma_N = [I, \{S_i\}, \{u_i(.)\}]$.*

Definition 6 *A pure strategy specifies a deterministic choice $s_i(h)$ at each information set $h \in H_i$.*

Definition 7 *Given player i's pure strategy set S_i, a mixed strategy for player i is $\sigma_i : S_i \to [0, 1]$ assigns to each pure strategy $s_i \in S_i$ a probability $\sigma_i(s_i) \geq 0$ that it will be played where $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$*

Definition 8 *A strategy $s_i \in S_i$ is a strictly dominant strategy for player i in game $\Gamma_N = [I, \{S_i\}, \{u_i(.)\}]$ if for all $s'_i \neq s_i$ we have*

$$u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$$

for all $s_{-i} \in S_{-i}$

Definition 9 *A strategy $s_i \in S_i$ is strictly dominated for player i in game $\Gamma_N = [I, \{S_i\}, \{u_i(.)\}]$ if there exists another strategy $s'_i \in S_i$ such that for all $s_{-i} \in S_{-i}$*

Definition 10 *In a game* $\Gamma_N = [I, \{S_i\}, \{u_i(.)\}]$*, a strategy* σ_i *is a best response for player* i *to his rivals' strategies* σ_{-i} *if*

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

for all $\sigma_i' \in \Delta(S_i)$.

Definition 11 *In game* $\Gamma_N = [I, \{S_i\}, \{u_i(.)\}]$*, the strategies in* $\Delta(S_i)$ *that survive the iterated removal of strategies that are never a best response are known as players* $i's$*, rationalizable strategies.*

Definition 12 *A strategy profile* $s = (s_1, ..., s_I)$ *constitutes a NE of the game* $\Gamma_N = [I, \{S_i\}, \{u_i(.)\}]$ *if for every* $i = 1, ..., I$,

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$

for all $s_i' \in S_i$

Definition 13 *A mixed strategy profile* $\sigma = (\sigma_1, ..., \sigma_I)$ *constitutes a MSNE of game* $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(.)\}]$ *if for every* $i = 1, ..., I$,

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

for all $\sigma_i' \in \Delta(S_i)$

Theorem 14 *Kakutani's Fixed Point Theorem: Let* X *be a compact convex subset of* $\mathbb{R}^n$ *and let* $f : X \to X$ *be a set valued function for which*

- *for all* $x \in X$ *the set* $f(x)$ *is nonempty and convex*
- X *is compact, convex and nonempty subset of a finite-dimensional Euclidean space*

Then there exists $x^* \in X$ *such that* $x^* \in f(x^*)$*. (p. 943 of Mas-Collel)*

Definition 15 $f : \mathbb{R}^n \to \mathbb{R}^n$ *is convex is* $f(\alpha x + (1-\alpha)x') \leq \alpha f(x) + (1-\alpha)f(x')$

Definition 16 *The set* $A \subset \mathbb{R}^n$ *is convex if* $\alpha x + (1-\alpha)x' \in A$ *whenever* $x, x' \in A$ *and* $\alpha \in [0, 1]$

Definition 17 *A subset* X *of a Euclidean space is compact if for any sequence in* X *has a subsequence that converges to a limit point in* X.

Proposition 18 *A Nash equilibrium exists in game* $\Gamma_N = [I, \{S_i\}, \{u_i(.)\}]$ *if for all* $i = 1, ...I$,

1. S_i *is nonempty, convex and compact subset of some Euclidean space* $\mathbb{R}^M$
2. $u_i(s_1, ..., s_I)$ *is continuous in* $(s_1, ..., s_I)$ *and quasiconcave in* s_i

Proof. Define the correspondence $B : S \to S$ by

$$B(s_1, ..., s_I) = b_1(s_{-1}) \times ... \times b_I(s_{-I})$$

Now to show a NE exists, we need to show that a FP exists as they are the same.

So we must show $B(s)$ is nonempty (given since $u(S)$ is continuous and S is compact), is convex (given by fact that u_i is quasiconcave in s_i) and has a closed graph (given by continuity of u_i) Hence, by Kakutanis' FPT, there exists a fixed point for this function and therefore a strategy profile $\widehat{s} \in S$ such that $\widehat{s} \in b(\widehat{s})$. The strategies at this fixed point constitute a NE because by construction $\widehat{s}_i \in b_i(\widehat{s}_{-i})$. See page 29 of Fudenberg and Tirole for more detail. ■

Definition 19 *quasiconcave: The function $f : A \to \mathbb{R}$ on the convex set $A \subset \mathbb{R}^n$ is quasiconcave if*

$$f(x) \geq t \text{ and } f(x') \geq t \text{ implies that } f(\alpha x + (1-\alpha)x') \geq t \tag{1}$$

for any $t \in \mathbb{R}$ and $x, x' \in A$ and $\alpha \in [0, 1]$

Definition 20 *A Bayesian game consists of*

- *A finite set of I players*
- *A finite set of states, Θ.*
- *A set of actions for each player, S_i*
- *a finite set T_i (the set of signals that may be observed by player i) and a function $\tau : \Theta \to T_i$ (signal function)*
- *a probability function, p, on Θ (the prior belief function) for which $p_i(\tau_i^{-1}(t_i)) > 0$ for all $t_i \in T_i$. So, given your signal what is your belief of the true state of the world.*
- *payoff function $u_i(s_i, s_{-i}, \theta_i)$*

A Bayesian game can be described by $[I, \{S_i\}, \{T_i\}, \{\tau_i\}.\{p_i\}\{u_i(.)\}, \Theta]$.

NOTE: definition in the book is slightly different. A special case where everyone observes same signal and has same prior, summarized by the function $F(\Theta)$. I like this one better.

Player $i's$ expected payoff given pure strategies for the I players is then

$$\widetilde{u}_i(s_1(.), ..., s_I(.)) = E_\theta[u_i(s_1(t_1), ..., s_I(t_I), t_i] \tag{2}$$

Definition 21 *A pure strategy BNE for the game $[I, \{S_i\}, \{u_i(.)\}, \Theta, F(.)]$ is a profile of decision rules $(s_1(t), ..., s_I(t))$ that constitutes a NE of the game $[I, \{\mathcal{S}_i\}, \{\widetilde{u}_i(.)\},]$ or for every $i = 1, ..., I$,*

$$\widetilde{u}_i(s_i(.), s_{-i}(.)) \geq \widetilde{u}_i(s_i'(.), s_{-i}(.))$$

for all $s_i' \in \mathcal{S}_i$

Definition 22 *A subgame of the game Γ_E is a subset of the game having the following properties*

- *it begins with an information set containing a single decision node, contains all the successors of this node and only those nodes*
- *if a decision node x is in the subgame then every $x' \in H(x)$ is also where $H(x)$ is the information set that contains x.*

Definition 23 *A profile of strategies $\sigma = (\sigma_1, ..., \sigma_I)$ is an I players extensive form game Γ_E is a subgame perfect NE if it induces a NE at every subgame of Γ_E*

Definition 24 *A System of beliefs μ in extensive form game Γ_E is a specification of a probability $\mu(x) \in [0,1]$ for each decision node x in Γ_E such that $\sum_{x \in H} \mu(x) = 1$ for all information sets in H.*

Let $E[u_i|H, \mu, \sigma_i, \sigma_{-i}]$ be the expected utility starting at information set H if beliefs are given by μ and if they follow σ_i with their opponents following σ_{-i}. Also let $\iota(H)$ be the player who moves at information set H then

Definition 25 *A strategy profile σ in extensive form game Γ_E is sequentially rational at information set H given a system of beliefs μ if we have*

$$E[u_{\iota(H)}|H, \mu, \sigma_{\iota(H)}, \sigma_{-\iota(H)}] \geq E[u_{\iota(H)}|H, \mu, \widetilde{\sigma}_{\iota(H)}, \sigma_{-\iota(H)}]$$

for all $\widetilde{\sigma}_{\iota(H)} \in \Delta(S_{\iota(H)})$. If a strategy profile σ satisfies this condition for all H then that σ is sequentially rational given belief system μ.

Baye's Rule

$$Prob(x|H, \sigma) = \frac{Prob(x|\sigma)}{\sum_{x' \in H} Prob(x'|\sigma)} \tag{3}$$

Definition 26 *A profile of strategies and a system of beliefs (σ, μ) is a wPBE in extensive form game Γ_E if it has the following properties*

1. *The strategy profile is sequentially rational given beliefs*
2. *The system of beliefs is derived from strategy profile σ through Bayes' rule whenever possible. Or for any information set H such that Prob($H|\sigma$) > 0 we must have*

   $$\mu(x) = \frac{Prob(x|\sigma)}{Prob(X|\sigma)} \quad \text{for all } x \in H$$

   or probability of being at a particular node in H is computed by the probability of reaching that node/probability of reaching all other nodes in x.

Definition 27 *A perfect Bayesian equilibrium consists of a strategy profile and a belief profile such that*

1. *The collection of strategies constitute a NE given the player's beliefs.*
2. *At each information set, the move required by the player's strategy maximizes the player's expected utility given the player's beliefs about the state of the world at that information set and other players' strategies*
3. *wherever possible, every player's beliefs can be derived from the equilibrium strategy profile and the common prior beliefs using Bayesian updating*

Definition 28 *Define player i's minmax payoff in game G, v_i, is the lowest players that other players can force upon player i*

$$v_i = \min_{s_{-i} \in S_i} \max_{s_i \in S_i} u_i(s_i, s_{-i}) \tag{4}$$

Definition 29 *A payoff profile w that delivers payoff level w_i to all i is said to be enforceable if $w_i \geq v_i$. Strictly enforceable if $w_i > v_i$ for all i.*