Notes on Inequality Constrained Optimization
ECO4401/5403: Introduction to Mathematical Economics/Static Optimization

9/21/2006

The general form of the problem we want to solve is

$$\begin{aligned} \max z &= f(x,y) \\ \text{st } g(x,y) &= b \\ h_1(x,y) &\le d_1 \\ h_2(x) &\le d_2 \end{aligned}$$

A couple of notes. First, just for convenience I'll be referring to equality constraint functions as $g(x,y)$ and inequality as $h(x,y)$. Just notation, nothing important. It is, however, important that the inequality constraints are written with the function being greater than some amount, not less than. If it is written as greater than, just multiply by -1. Let us rewrite the constraints to get them in their proper form

$$\begin{aligned} b - g(x,y) &= 0 \\ d_1 - h_1(x,y) &\ge 0 \\ d_2 - h_2(x)) &\ge 0 \end{aligned}$$

The Lagrangian for this is

$$\mathcal{L}(x,y,\lambda,\mu_1,\mu_2) = f(x,y) + \lambda(b - g(x,y)) + \mu_1(d_1 - h_1(x,y)) + \mu_2(d_2 - h_2(x))$$

The signs in front of your multipliers for the inequality constraints are important and tricky. Notice that we have the constraints written as the constraint being ≥ 0. If written as $\le$, you need - signs in the Lagrangian. Also, just as a convention, for inequality constraints I am using μ as the multiplier for them to differentiate them from the equality constraints where I use λ. I am using μ with different subscripts to distinguish the multiplier for different constraints. Just makes things clearer. You are free to use whatever symbols make sense to you.

In this version I have included three constraints mainly to show you how to deal with multiple constraints. I've done one of these where the function includes both variables and one with only one. As you will see, there are really no differences in the method.

Notice that our Lagrangian has 5 variables now which means five FOC's need to be taken. So.

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= f_x - \lambda g_x - \mu_1 h_{1x} - \mu_2 h_{2x} = 0 \\
\frac{\partial \mathcal{L}}{\partial y} &= f_y - \lambda g_y - \mu_1 h_{1y} = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= b - g(x, y) = 0 \\
\frac{\partial \mathcal{L}}{\partial \mu_1} &= d_1 - h_1(x, y) \geq 0 \\
\frac{\partial \mathcal{L}}{\partial \mu_2} &= d_2 - h_2(x) \geq 0
\end{aligned}$$

If all of our constraints were of the equality sort, we would just solve these five equations for our five unknowns and be done. The inequality constraints mean we need two additional types of conditions to replace these last two derivatives as these are actually not what we need. The first is that all of the multipliers for the inequality constraints must be ≥ 0 or

$$\begin{aligned}
\mu_1 &\geq 0 \\
\mu_2 &\geq 0
\end{aligned}$$

We also need some complementary slackness conditions for the inequality constraints which are .

$$\begin{aligned}
\mu_1(d_1 - h_1(x, y)) &= 0 \\
\mu_2(d_2 - h_2(x)) &= 0
\end{aligned}$$

These terms have to work out to be 0 so that at the optimum, again, our Lagrangian has the same value as our objective function. So the way to read these is to consider the two possibilities. One is that $d_1 - h_1(x, y) = 0$. In that case, it does not matter what our multiplier is (so long as it is positive) because our constraint is binding. Another possibility is that $d_1 - h_1(x, y) > 0$ which would mean that our constraint does not bind. In that case our multiplier must be zero.

Let's do a simple example:

$$\begin{aligned}
\max z &= 10x - x^2 + 180y - y^2 \\
\text{st } x + y &\leq 80 \\
x &\geq 0 \\
y &\geq 0
\end{aligned}$$

One thing you can do to see if the constraints matter is solve for the unconstrained max and ignore the 3 constraints. This takes 2 FOC's.

$$\begin{aligned}
\partial z / \partial x &= 10 - 2x = 0 \\
x &= 5 \\
\partial z / \partial y &= 180 - 2y = 0 \\
y &= 90
\end{aligned}$$

Our unconstrained max would be at $x + y = 95$. Notice that this is indeed larger than allowed for by our first constraint. So it will be binding. If we had found, say $x = 5$ and $y = 60$, then we could say we were done and not have to worry about the rest of the work involved in dealing with the constraints. Always worth a chance to check it out.

If we actually have to find the solution with the constraints, things are not as easy. Let's go through how to do it.

First thing to do is write down the Lagrangian being very careful with the +/- signs. Let us re write the constraints to make this easy.

$$\begin{aligned} 80 - x - y &\geq 0 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

So the Lagrangian is

$$\mathcal{L}(x, y, \mu_1, \mu_2, \mu_3) = 10x - x^2 + 180y - y^2 + \mu_1(80 - x - y) + \mu_2 x + \mu_3 y$$

The only two FOC's we really need are with respect to x and y.

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= 10 - 2x - \mu_1 + \mu_2 = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= 180 - 2y - \mu_1 + \mu_3 = 0 \end{aligned}$$

After that, we write down everything else we know must be true about our solution. The first three things we know must be true are our constraints. It is actually easier to think of the conditions for our constraints this way rather than as derivatives since this way we get the inequalities right.

$$\begin{aligned} 80 - x - y &\geq 0 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

The final things we know must be true are our complementary slackness conditions:

$$\begin{aligned} \mu_1(80 - x - y) &= 0 \\ \mu_2 x &= 0 \\ \mu_3 y &= 0 \\ \mu_1 &\geq 0 \\ \mu_2 &\geq 0 \\ \mu_3 &\geq 0 \end{aligned}$$

At this point we have a system of 11 equations and 5 unknowns which means that we should be able to solve for our answer. Unfortunately, doing this is not nearly as straightforward as before since we have more equations than unknowns. Looking for a solution is pretty much a matter of trial and error, but for the problems you will see in this class they should typically be pretty easy to find.

One of the best ways to proceed is to start playing a "what if game" with the constraints and complementary slackness conditions and see if we encounter any contradictions in our attempt to find a solution.

What if $y = 0$? Then from our first constraint, and complementary slackness condition we know that either $x = 80$ or $\mu_1 = 0$. Let us try $x = 80$. Our FOC for x tells us that $10 - 2x - \mu_1 + \mu_2 = 0$, substituting in $x = 80$ gives us $-150 - \mu_1 + \mu_2 = 0$ or $\mu_2 - \mu_1 = 150$

We also know that $180 - 2y - \mu_1 + \mu_3 = 0$ and since $y = 0$, we have $180 = \mu_1 - \mu_3$

Since all $\mu' s \geq 0$, and also that we must have $\mu_1 \geq 180$. We previously showed that $\mu_2 - \mu_1 = 150$ and so $\mu_2 \geq 150 + 180 = 330$. Since we have assumed that $x = 80$, and our complementary slackness condition of $\mu_2 x = 0$ must hold this means $\mu_2 = 0$. This gives us a contradiction as μ_2 can not be simultaneously 0 and ≥ 330. This ultimately tells us that we can not find a solution where $y = 0$ and/or $x = 80$.

Let us try another obvious one, perhaps $x = 0$. If $x = 0$ then $y = 80$ or $\mu_1 = 0$. Let us try $y = 80$. Then $\mu_3 = 0$.

Also since $180 - 2y - \mu_1 + \mu_3 = 0$, we have that $180 - 160 - \mu_1 + 0 = 0$ or $\mu_1 = 20$.

Since $10 - 2x - \mu_1 + \mu_2$, we have that $10 - 0 - 20 + \mu_2 = 0$ or $\mu_2 = 10$.

This gives us solutions for all of our variables with no contradictions. So $x = 0, y = 80, \mu_3 = 0, \mu_2 = 10, \mu_1 = 20$ and we are done.

The way to solve a CO problem with inequality constraints is to just try some solutions until you find one without any contradictions. As you can imagine, this is typically a pain, but you should not have many that are all that difficult.[1] If the only inequality constraints are that $x, y \geq 0$ then I suggest solving the problems first ignoring that constraint and only when you find it violated do you decide to try this.

Let us try another to illustrate a few more points

$$\begin{aligned} \max z &= 100x - 5x^2 + 500y - 5y^2 \\ \text{st} \quad & 10x + 10y \leq c \\ & x, y \geq 0 \end{aligned}$$

Full Lagrangian is

$$\mathcal{L} = 100x - 5x^2 + 500y - 5y^2 + \mu_1(c - 10x - 10y) + \mu_1 x + \mu_3 y$$

Notice that I have not indicated a specific value for c yet. I want to let it be vague at the moment so that I can show you the different types of solutions we will get depending on what value I plug in there. If we wanted to solve this we would have to go through all of our conditions which are:

$\frac{\partial \mathcal{L}}{\partial x} = 0$

$\frac{\partial \mathcal{L}}{\partial y} = 0$

3 constraints, comp slackness conditions and $\mu_1, \mu_2, \mu_3 \geq 0$

There are a couple of simpler ways to solve this, when they work.

The first approach would be to solve this as an unconstrained problem, ignoring all of our constraints. This is equivalent to the what if game of $\mu_1, \mu_2, \mu_3 = 0$. If we do that and find that none of the three constraints are violated then we are done. Let's give that a shot:

[1] There are other approaches that are more computationally efficient in this respect that do not involve going through these conditional arguments. If you know these approaches already, you are free to use them. I prefer to teach this way as I think it helps you get a more intuitive understanding of the process.

FOC: $100 - 10x = 0 \quad ==> x = 10$

$500 - 10y = 0 ==> y = 50$

Now we check to see if our constraint is violated: $10(10) + 10(50) = 600$

If $600 \leq c$ then we are fine, if not then we have to solve the full problem.

Let us assume that $c < 600$.

We have to solve a bigger problem but there is another legitimate assumption. See the complementary slackness condition that $\mu_1(c - 10x - 10y) = 0$. Well, the game we just played was assume $\mu_1 = 0$. Now let us try $(c - 10x - 10y) = 0$. That means solving our problem as an equality constrained problem but we will keep $\mu_2, \mu_3 = 0$. So

$\mathcal{L} = 100x - 5x^2 + 500y - 5y^2 + \mu_1(c - 10x - 10y)$

FOC's: $100 - 10x - 10\mu_1 = 0$

$500 - 10y - 10\mu_1 = 0$

$c - 10x - 10y = 0$

Then we solve first 2 equations for μ_1, set them equal and get $10 - x = 50 - y$ or $x = -40 + y$.

Sub that into our constraint or third FOC and get $c - 10(-40 + y) - 10y = 0$, Solution is: $\left\{y = \frac{1}{20}c + 20\right\}$

$x = -40 + \frac{1}{20}c + 20 = -20 + \frac{1}{20}c$.

When can we accept this as a solution? So long as $x,y \geq 0$ which holds when $x = -20 + \frac{1}{20}c \geq 0$, Solution is: $\{400 \leq c\}$

We also have to verify that $\mu_1 \geq 0$. Well for that we need that $10 - x = 10 + 20 - \frac{1}{20}c \geq 0$

$10 + 20 - \frac{1}{20}c \geq 0$, Solution is: $\{c \leq 600\}$

So let us say $c = 500$, then I would have

$y = \frac{1}{20}(500) + 20 = 45$

$x = -20 + \frac{1}{20}(500) = 5$

$\mu_1 = \frac{100 - 5(5)}{10} = \frac{15}{2}$

And we are set.

Let us assume however that we have the problem.

$$\max z = 100x - 5x^2 + 500y - 5y^2$$
$$\text{st} \quad 10x + 10y \leq 40$$
$$x,y \geq 0$$

Now, what we see is that the value of c is such that we can not solve this with either approach. As the values we found above, $x = 10$, $y = 50$ do not satisfy our constraint now. Also, we can not assume that our constraint holds as an equality as doing so leads to a violation of our non-negativity constraints. For this we will need to solve the full problem:

$\mathcal{L} = 100x - 5x^2 + 500y - 5y^2 + \mu_1(40 - 10x + 10y) + \mu_2 x + \mu_3 y$

Let's see how the solution works out:

Full conditions are:

$\frac{\partial \mathcal{L}}{\partial x} = 100 - 10x - 10\mu_1 + \mu_2 = 0$
$\frac{\partial \mathcal{L}}{\partial y} = 500 - 10y - 10\mu_1 + \mu_3 = 0$
$10x + 10y \leq 40$
$x \geq 0$
$y \geq 0$
$\mu_1(10x + 10y - 40) = 0$
$\mu_2 x = 0$
$\mu_3 y = 0$
$\mu_1 \geq 0$
$\mu_2 \geq 0$
$\mu_3 \geq 0$

To solve this we have to play our what if games. Lets try $x = 0$.

In that case $100 - 10x - 10\mu_1 + \mu_2 = 0$, or $100 + \mu_2 = 10\mu_1$. No contradiction there, both are positive, so long as $\mu_1 > 10$.

If $\mu_1 > 10$, then from our first complementary slackness conditions we must have $40 - 10x - 10y = 0$ or $40 = 10y$ or $y = 4$. It is positive so no problem.

Since $y > 0$, we must have $\mu_3 = 0$ and all of our comp slackness conditions are satisfied. To get our last variables, take the FOC for y and:

$500 - 10y - 10\mu_1 + \mu_3 = 0$ or $500 - 10(4) - 10\mu_1 = 0$, Solution is: $\{\mu_1 = 46\}$

Therefore: $100 + \mu_2 = 10\mu_1$, or $100 + \mu_2 = 10(46)$, Solution is: $\{\mu_2 = 360\}$

Our function is $z = 100x - 5x^2 + 500y - 5y^2 = 500(4) - 5(4)^2 = 1920$

And we have a complete answer:

$x^* = 0$	$y^* = 4$	$\mu_1^* = 46$
$\mu_2^* = 360$	$\mu_3^* = 0$	$z^* = 1920$