Notes on Principal Agent Problem

Tim Salmon

October 2006

Proposition 1 *In the PA problem with unobservable effort and agent with RN preferences, an optimal contract generates the same effort choice and expected utilities for agent and principal as when effort is observable.*

Proof. In the unobservable case, we need to solve problem for both P and A. P gets to choose w, and then A gets to choose the corresponding e. This is problem of the Principal

$$\max_{w} E[R(e)] - E[w]$$
$$st \;\; E[w] - c(e) \geq \overline{\mu}$$

Why can I drop the second constraint? Because I'm about to solve the agent problem separately. Before I just put them together. Let us propose setting $w_i = R_i - \alpha$ and therefore $E[w] = E[R] - \alpha$ then the problem becomes

$$\max_{\alpha} E[R(e)] - (E[R(e)] - \alpha) = \alpha$$
$$st \;\; E[R(e)] - \alpha - c(e) \geq \overline{\mu}$$

Obvious solution is $\alpha^* = E[R(e)] - c(e) - \overline{\mu}$,

We then get e^*, from agents problem

$$\max_{e} E[w] - c(e) = \max_{e} E[R(e)] - \alpha - c(e)$$
$$st \;\; E[w] - c(e) = E[R(e)] - \alpha - c(e) \geq \overline{\mu}$$

First notice that agents incentives are now to find e^* that maximizes R. And if we have $\alpha^* = E[R(e^*)] - c(e^*) - \overline{\mu}$, we know constraint is binding (plug this into constraint and it reduces to $\overline{\mu} \geq \overline{\mu}$). Therefore $E[R(e^*)] - \alpha - c(e^*) = \overline{\mu}$, or $E[R(e^*)] - \alpha = E[w] = c(e^*) + \overline{\mu}$. So the expected wage is equal to the cost of the effort choice plus reservation utility which is also equal to $E[R(e^*)] - \alpha$.

What about observable effort? The principal's problem becomes

$$\max_{e} E[R(e)] - w(e)$$
$$st \; w(e) \geq c(e) + \overline{\mu}$$

Revenue, R, is still uncertain here because the outcome of the agents action is still uncertain, only difference is that principal can actually observe effort in addition to outcome. If he can observe effort then he essentially gets to choose the effort level himself as since he knows how the agent will respond to any wage profile, he can set the wage profile up to induce an effort choice and essentially choose effort. We are therefore incorporating the agents problem into the principals problem and do not have to solve it separately. Obvious again that constraint binds, so problem can be rewritten as through substitution as

$$\max_{e} E[R(e)] - c(e) - \overline{\mu}$$

Look closely at this problem and the one of the agent before. In both cases we are trying to max $E[R(e)] - c(e) - K$ where K is some constant, either α or $\overline{\mu}$. Consequently, the same e^* solves both. Therefore effort levels are the same in the unobservable as in the observable effort cases.

What about wage levels? in observable $w^* = c(e^*) + \overline{\mu}$, in unobservable case $E[w^*] = E[R(e^*)] - \alpha = c(e^*) + \overline{\mu}$. Since e^* is the same, these are the same numbers.

Finally since e^* is the same across both cases, $E[R(e^*)]$ is as well. Combine this with w^* being the same and we have the profits to the principal being the same across both cases since they are $E[R(e^*)] - E[w^*]$ in the uncertain information case and $E[R(e^*)] - w^*$ in the certain info case. ■

Case 2: Risk Averse Agent

Let us go back to our original agent or $u(w, a) = \sqrt{w} - e$, $\overline{u} = 9$, reservation utility, $e_h = 5$ and $e_l = 0$

and keep the same info on our principal, RN and

$e = h$ ==>$p_{400} = .6$, $p_{100} = .3$, $p_0 = .1$

$e = l$ ==>$p_{400} = .1$, $p_{100} = .3$, $p_0 = .6$

recall that the perfect info contract was ($196,0) and expected profit is 270-196=74.

Proposition 2 *In the PA game with a RA agent, it is not possible to set up a contract that will yield the efficient outcome.*

Proof. It is an easy proof that if one party is RA and one RN then one who is RN will bear all of the risk in the efficient eq. The reason is that the RA party does not want risk and will always be willing to pay the RN person a little bit to remove it. Since the RN person only cares about the expected value, he doesn't care one way or the other so he takes all of the risk. See the insurance game we covered earlier. In this case, that would involve setting a single wage of $196. We have already seen that this will not be sustained as an eq. In order to get our agent to bear any risk, we are going to have to pay them more than $196 in expectation. Since they are RA, however and the principle is RN, there would always exist a trade in which both would be better off by trading some risk back to the principle, except for the problem that this would have the agent no longer choosing the appropriate effort level.

Another way to look at this is that if we induce our agent to engage in effort, we must pay them more than an expected $196. Why? Because they could essentially get that for certain by doing nothing. In order to get them to do work, you must pay them more in order to shoulder some of the risk. This will be inefficient. ■

To solve this problem, we just need to do exactly what we did before though. Slight redefinition for convenience though $w_{400} = g^2$ etc. . . so assuming that principal wants to elicit high effort, problem is

$$\min_{b,m,g} E[w] = .1b^2 + .3m^2 + .6g^2 \tag{1}$$

subject to

$$.1b + .3m + .6g - 5 \geq 9 \tag{2}$$

$$.1b + .3m + .6g - 5 \geq .6b + .3m + .1g \tag{3}$$

Way to do this one is:

$L = .1b^2 + .3m^2 + .6g^2 + \lambda(9 - (.1b + .3m + .6g - 5))+$

$\mu(.6b + .3m + .1g - (.1b + .3m + .6g - 5))$

Conditions:

$\frac{dL}{dg} = 1.2g + \lambda(-.6) + \mu(.1 - .6) = 1.2g - 0.6\lambda - 0.5\mu = 0$

$\frac{dL}{dm} = .6m + \lambda(-.3) + \mu(.3 - .3) = 0.6m - 0.3\lambda = 0$

$\frac{dL}{db} = .2b + \lambda(-.1) + \mu(.6 - .1) = 0.2b - 0.1\lambda + 0.5\mu = 0$

$.1b + .3m + .6g - 5 \geq 9$
$\lambda(9 - (.1b + .3m + .6g - 5)) = 0$
$.1b + .3m + .6g - 5 \geq .6b + .3m + .1g$
$\mu(.6b + .3m + .1g - (.1b + .3m + .6g - 5)) = 0$
$\lambda, \mu \geq 0$

Notice, there are no non-neg conditions on wages.
$\frac{dL}{dg}$: $1.2g - 0.6\lambda - 0.5\mu = 0$, Solution is: $\{g = (0.5\lambda + 0.41667\mu)\}$
$\frac{dL}{dm}$: $0.6m - 0.3\lambda = 0$, Solution is: $\{m = (0.5\lambda)\}$
$\frac{dL}{db}$: $0.2b - 0.1\lambda + 0.5\mu = 0$, Solution is: $\{b = (0.5\lambda - 2.5\mu)\}$

Assume that $\lambda > 0$ then $9 - (.1b + .3m + .6g - 5) = 0$ or
$9 - (.1(0.5\lambda - 2.5\mu) + .3(0.5\lambda) + .6(0.5\lambda + 0.41667\mu) - 5) = 0$, Solution is: $\left\{\lambda = \left(28.0 - 4.0 \times 10^{-6}\mu\right)\right\}$
plugging in to solutions for wages. . .
$g = \left(0.5\left(28.0 - 4.0 \times 10^{-6}\mu\right) + 0.41667\mu\right) = (14.0 + 0.41667\mu)$
$m = \left(0.5\left(28.0 - 4.0 \times 10^{-6}\mu\right)\right) = \left(14.0 - 2.0 \times 10^{-6}\mu\right)$
$b = \left(0.5\left(28.0 - 4.0 \times 10^{-6}\mu\right) - 2.5\mu\right) = (14.0 - 2.5\mu)$

We must assume that μ >0 or all wages are not the same (pretty reasonable assumption):
So $.6(14.0 - 2.5\mu) + .3\left(14.0 - 2.0 \times 10^{-6}\mu\right) +$
$.1(14.0 + 0.41667\mu) - .1(14.0 - 2.5\mu) +$
$.3\left(14.0 - 2.0 \times 10^{-6}\mu\right) + .6(14.0 + 0.41667\mu) - 5) = 0$, Solution is: $\{\mu = 3.4286\}$

So $g = (14.0 + 0.41667(3.4286)) = 15.429$
$m = \left(0.5\left(28.0 - 4.0 \times 10^{-6}\mu\right)\right) = \left(14.0 - 2.0 \times 10^{-6}(3.4286)\right) = 14.000$
$b = \left(0.5\left(28.0 - 4.0 \times 10^{-6}\mu\right) - 2.5\mu\right) = (14.0 - 2.5(3.4286)) = 5.4285$

Result is $b = 5.42587$, $m = 14$ and $g = 15.42857$. or $w = \{238.0407, 196, 29.46\}$ Leads to
$E[w] = 204.58 > 196$ which was wage in Obs case. and $E[\pi] = 270 - 204.58 = 65.42 < 74$ from perf info case

So due to the fact that effort level of agent is not observable, the principal must pay the agent an extra \$8.58 for agent to shoulder part of the risk. That is, if the principal wants to induce $e = h$. If the principal wants to induce $e = l$ then this is not the case. Principal can just pay reservation wage and be done with it.