# Contests between groups of unknown size

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**Abstract**

We study group contests where the number of competing groups is fixed but group sizes are stochastic and unobservable to contest participants at the time of investment. We allow for arbitrary correlations in the joint distribution of group sizes. When the distribution is symmetric, with expected group size $\bar{k}$, the symmetric equilibrium aggregate investment is lower than in a symmetric group contest where the group size is fixed and commonly known to be $\bar{k}$. A similar result holds for asymmetric distributions of group sizes in contests between two groups. For the symmetric case, the reduction in individual and aggregate investment due to group size uncertainty is stronger the larger the variance in appropriately defined relative group impacts. When group sizes are independent conditional on a common shock, a stochastic increase in the common shock mitigates the effect of group size uncertainty unless the common and idiosyncratic components of group size are strong complements. We also show that the introduction of group size uncertainty undermines the robustness of the group size paradox otherwise present in the model.

*Keywords*: group contest, stochastic group size, population uncertainty, relative group impact, group size paradox
*JEL codes*: C72, D72, D82

## 1 Introduction

In this paper, we study group contests where the number of competing groups is fixed but group sizes are stochastic and unknown to participants at the time of investment. We consider the simplest, canonical group contest setting in which the prize awarded to the winning group is non-rival and the efforts of individuals are perfect substitutes within groups. Group sizes are determined stochastically according to a joint distribution, where we allow for arbitrary correlations between group sizes, including those driven by a common shock.

Contests arise in a wide range of settings. Examples include competition for bonuses and promotions in organizations, rent-seeking activities, such as companies fighting for government contracts, or lobbyists promoting legislation, as well as litigation, political campaigns and R&D

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competition (for related reviews see, e.g., Lazear, 1999; Congleton, Hillman and Konrad, 2008). In many cases, these contests involve *groups* of independent actors competing for a common goal in order to secure a non-rival prize for all members of the winning group. For example, loosely defined groups of US telecommunication giants (such as Comcast and Verizon) and Internet content providers (the likes of Netflix, Amazon, Microsoft, Google and Facebook) find themselves on opposite sides of the net neutrality debate and lobby actively for their respective interests. Furthermore, in many of these settings, the number of players in each of the competing groups may not be exactly known, especially as many companies refrain from taking a public stance on the issue, while working covertly behind the scenes. The same applies to political campaigns where, especially after the US Supreme Court's Citizens United vs. Federal Election Commission decision, the number and identity of donors is easy to conceal.

More recently, researchers have challenged the standard assumption in the contest literature that the number of competitors is commonly known, thereby exploring the effects of population uncertainty on behavior in contests between individuals (Münster, 2006; Myerson and Wärneryd, 2006; Lim and Matros, 2009; Fu, Jiao and Lu, 2011; Kahana and Klunover, 2015, 2016; Ryvkin and Drugov, 2017; Boosey, Brookins and Ryvkin, 2017).[1] However, to the best of our knowledge, there is no study to date that examines the effects of population uncertainty on behavior in group contests.[2] To the extent that the uncertainty relates to the number of groups, many of the insights from contests between individuals can be naturally extended. However, there is another dimension – uncertainty about the *sizes* of the groups – through which population uncertainty may operate in group contests. In contrast to the existing literature on games with population uncertainty, such a setting is more similar to private information in Bayesian games, where the incomplete information relates to players' types. Intuitively, a player who is active in the group contest updates her beliefs about the size (and hence the relative "strength") of her own group and, in case the group sizes are correlated, about the sizes of other groups.

By allowing for arbitrary correlations between group sizes, our model can also accommodate common shocks to group sizes. The way that contest participants respond to population uncertainty or to the correlation (positive and negative) induced by various structural or policy-induced shocks is important for a wide range of prominent social and political issues. For example, the results of the 2016 Presidential election in the United States and ensuing changes in the regulatory climate have created a common shock to the sizes of groups fighting on different sides of many contentious issues such as net neutrality, health care, and environmental regulation.

[1]Similarly, population uncertainty has been studied theoretically in auctions (McAfee and McMillan, 1987; Harstad, Kagel and Levin, 1990; Levin and Ozdenoren, 2004) and other environments, such as voting, coordination games and public goods (e.g., Myerson, 1998, 2000; Makris, 2008, 2009; De Sinopoli and Pimienta, 2009; Mohlin, Östling and Wang, 2015).

[2]There is a well-developed theoretical literature on group contests examining a wide range of environments with common knowledge about group sizes (e.g., Katz, Nitzan and Rosenberg, 1990; Nitzan, 1991; Baik, 1993; Riaz, Shogren and Johnson, 1995; Nti, 1998; Esteban and Ray, 2001; Baik, 2008; Nitzan and Ueda, 2009, 2011; Ryvkin, 2011; Lee, 2012; Chowdhury, Lee and Sheremeta, 2013; Kolmar and Rommeswinkel, 2013; Brookins and Ryvkin, 2016; Barbieri and Malueg, 2016).

We characterize the semi-symmetric equilibrium in pure strategies for a general model with $n$ groups and an arbitrary joint distribution over group sizes, then provide three main results under additional assumptions. Our first main result illustrates that for symmetric distributions of group sizes, the symmetric equilibrium investment for any non-degenerate distribution with mean group sizes $\bar{k}$ is strictly lower than in a group contest where group sizes are fixed and commonly known to be $\bar{k}$. That is, population uncertainty (in terms of the sizes of the groups) lowers both the individual and aggregate equilibrium investment. This result is similar to findings regarding the effects of population uncertainty in *individual* contests derived by Myerson and Wärneryd (2006) and Lim and Matros (2009).

In particular, we show that the reduction in equilibrium investment increases with the variance of *relative group impact*, which is essentially the group's equilibrium probability of winning conditional on realized group sizes. As such, greater variance in relative group impact implies greater variance in the probability of winning, which reduces the marginal effect of investment on winning and hence leads to a reduction in equilibrium investment. This effect is similar to the (negative) impact of noise, or uncertainty in the winner determination process, on equilibrium effort in contests and tournaments. Alternatively, the notion of relative group impact can be interpreted as a (random) measure of the group's relative strength in the contest, in which case our result implies that the reduction in equilibrium investment is stronger when the distribution generates higher variance in the groups' relative strengths. This characterization echoes results on the effects of different kinds of player heterogeneity on aggregate effort in contests where it is generally demonstrated that larger asymmetries lead to lower effort.[3]

We also show that when group sizes are symmetrically distributed and correlated so that the size of each group is an increasing function of a common component and an idiosyncratic component, an increase in the common component (in the usual stochastic order) leads to a reduction in the variance of relative group impact unless the common and idiosyncratic components are strong complements. Thus, in most cases a large positive shock to all group sizes mitigates the effect of population uncertainty on individual and aggregate investment in contests.

Finally, we consider the case of two competing groups with (possibly) asymmetrically distributed group sizes with means $\bar{k}_1$ and $\bar{k}_2$. We show that, as for the symmetric distribution case, aggregate equilibrium investment is lower under population uncertainty than in the corresponding contest where the group sizes are fixed and commonly known to be $\bar{k}_1$ and $\bar{k}_2$. We also consider the effect of a stochastic increase in the size of a group on its probability of winning and show that it can be positive or negative depending on the details of the group size distribution. Thus, group size uncertainty is an additional channel undermining the robustness of the group size paradox (Olson, 1965; Esteban and Ray, 2001).

*Relation to previous literature*

Our model provides a novel extension to the existing theoretical literature on group contests.

[3] See, however, Ryvkin (2013) and Drugov and Ryvkin (2017) who show that this "common wisdom" is far from universal.

The natural complete information benchmark for our setting is provided by Baik (1993). He shows that only the highest-valuation players in each group will make positive investments, such that the total group effort is independent of group size. Moreover, even in a setting with "partial" group size uncertainty where each player knows the size of her own group but not the sizes of the other groups, this neutrality with respect to group size is readily extended. However, in our setting, where players also do not know the size of their own group, we show that population uncertainty has an effect on individual and aggregate investment.

Our study is also related to the existing literature on *individual contests* with a stochastic number of players. For lottery-type contests, Myerson and Wärneryd (2006) show that aggregate equilibrium investment in an uncertain contest with mean number of players equal to $\mu$ is strictly lower than in a contest where the number of players is equal to $\mu$ with certainty. Lim and Matros (2009) and Münster (2006) consider similar environments in which the number of players is a random variable drawn from the binomial distribution with parameters $(n, q)$ and also find that aggregate equilibrium investment is lower than in a corresponding contest with certain group size, $nq$. Ryvkin and Drugov (2017) generalize these results to a more general tournament model, also allowing for arbitrary distributions of the number of players. These findings are similar to our first main result concerning the negative effects of group size uncertainty on individual and aggregate equilibrium investment in the group contest setting.

The rest of the paper is organized as follows. In Section 2, we introduce the model and characterize the equilibrium. Section 3 provides general results and examples for symmetric distributions of group sizes. In Section 4, we consider contests between two groups and allow for arbitrary (possibly asymmetric) distributions of group sizes, then provide two examples to illustrate the results. Section 5 provides some concluding remarks.

# 2 The model

## 2.1 Preliminaries

Consider a contest between $n \geq 2$ groups indexed by $i = 1, \ldots, n$. The number of players in each group $i$, denoted by $K_i$, is a random variable drawn from the set $M_i = \{1, 2, \ldots, m_i\}$, where $m_i \geq 1$ can be finite or infinite.[4] We will use $\boldsymbol{K} = (K_1, \ldots, K_n)$ to denote the random vector of group sizes with support $M = \times_{i=1}^{n} M_i$, and $\boldsymbol{k} \in M$ to denote a generic realization of $\boldsymbol{K}$. Let $\mathbf{1} \in M$ denote a vector of ones, and $\geq$ denote the usual component-wise partial order (when applied to vectors). We use $p_{\boldsymbol{k}} = \Pr(\boldsymbol{K} = \boldsymbol{k})$ to denote the joint probability mass function

[4] We assume that the minimal number of players in each group is one; that is, we do not consider a scenario in which some groups may not exist at all. Such a setting would conflate the effects of uncertainty with respect to group sizes with those of uncertainty about the number of competing units. The latter effects have been explored extensively in the literature on individual contests with group size uncertainty (see, e.g., Münster, 2006; Myerson and Wärneryd, 2006; Lim and Matros, 2009; Ryvkin and Drugov, 2017). In this paper, we focus on the effects of group size uncertainty in group contests and keep the number of groups fixed.

(pmf) of group sizes. We allow for the possibility that group sizes are not independent; that is, $p_{\boldsymbol{k}}$ is not necessarily equal to the product of marginal pmfs $p_{k_i}$.

All participating players simultaneously choose investment levels $x_{ij} \in \mathbb{R}_+$, where $x_{ij}$ denotes the investment of player $j$ in group $i$ (referred to as "player $ij$"). The total investment of group $i$, denoted by $X_i$, is the sum of individual investments: $X_i = \sum_{j=1}^{K_i} x_{ij}$.

We consider a group contest with the contest success function (CSF) of the lottery form (Tullock, 1980) where each group's impact function is homogeneous of degree $r \in (0, 1]$. Thus, the probability that group $i$ wins the contest conditional on $\boldsymbol{K}$ is given by

$$P_i(X_i, X_{-i}|\boldsymbol{K} = \boldsymbol{k}) = \begin{cases} \frac{1}{n}, & \text{if } X_1 = \ldots = X_n = 0 \\ \frac{X_i^r}{\sum\limits_{l=1}^{n} X_l^r}, & \text{otherwise.} \end{cases} \quad (1)$$

All players in the winning group receive a prize normalized to one, while players in other groups receive zero prize. All players are risk-neutral expected payoff maximizers.

### 2.2 Equilibrium investment

In our setting, participating players do not observe the realization of $\boldsymbol{K}$ at the time of investment. From an outsider's perspective, $\sum_{\boldsymbol{k} \in M} p_{\boldsymbol{k}} \phi(\boldsymbol{k})$ then gives the expectation of some function $\phi(\boldsymbol{k})$ with respect to the joint distribution of group sizes. From the perspective of a participating player, however, the distribution of the vector of group sizes is updated (cf., e.g., Harstad, Kagel and Levin, 1990; Myerson and Wärneryd, 2006).

Let $I_{ij}$ denote a random variable equal to 1 if player $j$ is selected to participate in the contest as a member of group $i$, and equal to zero otherwise. Using Bayes' rule, player $ij$ should update the probability of any vector of group sizes $\boldsymbol{k}$ to

$$\tilde{p}_{\boldsymbol{k}}^i \equiv \Pr(\boldsymbol{K} = \boldsymbol{k}|I_{ij} = 1) = \frac{\Pr(I_{ij} = 1|\boldsymbol{K} = \boldsymbol{k})p_{\boldsymbol{k}}}{\sum_{\boldsymbol{l} \in M} \Pr(I_{ij} = 1|\boldsymbol{K} = \boldsymbol{l})p_{\boldsymbol{l}}}.$$

Assuming that players are equally likely to be selected as participants, it follows that

$$\tilde{p}_{\boldsymbol{k}}^i = \frac{k_i p_{\boldsymbol{k}}}{\bar{k}_i}, \quad (2)$$

where $\bar{k}_i = \sum_{\boldsymbol{l} \in M} l_i p_{\boldsymbol{l}}$ is the (prior) expected number of players in group $i$.[5]

It then follows from (2) that from the perspective of a participating player in group $i$ expectations are updated as $\frac{1}{\bar{k}_i} \sum_{\boldsymbol{k} \in M} p_{\boldsymbol{k}} k_i \phi(\boldsymbol{k})$. The payoff of player $ij$, conditional on being

[5]Formally, we model the selection process as follows. Consider the probability that $I_{ij} = 1$ given $\boldsymbol{K} = \boldsymbol{k}$. First, player $j$ is equally likely to be selected as one of the total number of participants, $\sum_{h=1}^{n} k_h$. Thus, the probability that $j$ is a participant at all is $\frac{\sum_{h=1}^{n} k_h}{\sum_{h=1}^{n} m_h}$. Second, the probability that player $j$ is a member of group $i$, given that she is a participant at all is equal to the number of players in group $i$, divided by the total number of participants, $\frac{k_i}{\sum_{h=1}^{n} k_h}$. Together, these observations imply that $\Pr(I_{ij} = 1|\boldsymbol{K} = \boldsymbol{k}) = \frac{k_i}{\sum_{h=1}^{n} m_h}$.

selected, is, therefore,

$$\pi_{ij} = \frac{1}{\bar{k}_i} \sum_{\boldsymbol{k} \in M} p_{\boldsymbol{k}} \frac{k_i X_i^r}{\sum_{h=1}^n X_h^r} - x_{ij}. \tag{3}$$

When group sizes are deterministic, this game has multiple (pure strategy Nash) equilibria such that the sum of investments in each group is given by the equilibrium investment in the corresponding contest between $n$ individuals (Baik, 2008). The reason is that all players in a group have the same marginal payoff that only depends on the group's aggregate investment; therefore, it does not matter how aggregate investment is allocated across players. In contrast, in the stochastic group size case any equilibrium is *semi-symmetric*; that is, all players within a group have the same investment level. To understand why, recall that, conditional on a realized group size, all players in the group have equal chances of being selected. Therefore, if players commit to different investment levels, the aggregate group investment will depend not only on the realized group size but also on which players are selected; however, in that case different players will have different marginal payoffs, which is impossible. The result is formalized in the following lemma (all missing proofs are relegated to Appendix A.).

**Lemma 1** *Suppose $K_i$, the number of players in group $i$, is non-degenerate. Then in any equilibrium all active players in group $i$ choose the same investment.*

In view of Lemma 1, in what follows we focus on the properties of a semi-symmetric equilibrium, where all players in group $i$ choose the same investment $x_i^*$. Assuming all participating players other than $ij$ choose such investment levels, the payoff of player $ij$ from some deviation investment $x_{ij}$ is

$$\pi_{ij}(x_{ij}, (x_h^*)_{h=1}^n) = \frac{1}{\bar{k}_i} \sum_{\boldsymbol{k} \in M} p_{\boldsymbol{k}} \frac{k_i(x_{ij} + (k_i - 1)x_i^*)^r}{(x_{ij} + (k_i - 1)x_i^*)^r + \sum_{h \neq i}(k_h x_h^*)^r} - x_{ij}. \tag{4}$$

The first-order conditions $\frac{\partial \pi_{ij}}{\partial x_{ij}} = 0$ evaluated at $x_{ij} = x_i^*$ produce the system of equations

$$\frac{r}{\bar{k}_i} \sum_{\boldsymbol{k} \in M} p_{\boldsymbol{k}} \frac{k_i^r (x_i^*)^{r-1} \sum_{h \neq i}(k_h x_h^*)^r}{(\sum_{h=1}^n (k_h x_h^*)^r)^2} = 1, \quad i = 1, \dots, n. \tag{5}$$

**Lemma 2** *For $r \in (0, 1]$, if $(x_i^*)_{i=1}^n$ is an interior solution to the system of equations (5), then it is a semi-symmetric equilibrium.*

Lemma 2 implies equilibrium existence (by construction) for situations when the existence of an interior solution to (5) can be established. More generally, the existence of equilibrium follows from Theorem 1 of Baye, Tian and Zhou (1993) that provides sufficient conditions for the existence of pure strategy equilibria in games with possibly discontinuous and non-quasiconcave payoffs.[6] This result, along with Lemmas 1 and 2, implies that the group contest with stochastic

[6]Theorem 1 of Baye, Tian and Zhou (1993) requires that (i) strategy spaces are nonempty, convex and compact

group sizes has a semi-symmetric equilibrium given by a solution of (5). Moreover, as long as group sizes are non-degenerate, any equilibrium has this structure.

Below we derive and study the solution to (5) (and hence, the equilibrium) for two important and relevant cases. First, we derive the fully symmetric equilibrium investment level for arbitrary $n$ under the assumption that the distribution of group sizes is symmetric. Second, we derive the semi-symmetric equilibrium investment levels for arbitrary (possibly asymmetric) distributions in the case where there are $n = 2$ groups.

## 3 Symmetric group size distributions

Consider the case where the distribution of group sizes is symmetric, i.e., $p_{\boldsymbol{k}} = p_{\rho(\boldsymbol{k})}$ for any permutation $\rho$ of the components of $\boldsymbol{k}$. In this case, we look for a fully symmetric equilibrium, with $x_i^* = x^*$ for all $i = 1, \dots, n$. Let $\bar{k} = \bar{k}_i$ denote the symmetric expected group size, and $m = m_i$ the symmetric maximum group size. The system of Eqns. (5) simplifies to

$$x^* = \frac{r}{\bar{k}} \sum_{\boldsymbol{k}} p_{\boldsymbol{k}} \frac{k_i^r \sum_{h \neq i} k_h^r}{(\sum_{h=1}^n k_h^r)^2}. \tag{6}$$

Let $S_i = \frac{K_i^r}{\sum_{h=1}^n K_h^r}$ denote the (random) *relative impact* of group $i$. Due to the symmetry of the joint distribution, since $\sum_{i=1}^n S_i = 1$, the expected relative impact of each group is $\mathrm{E}[S_i] = \frac{1}{n}$. Moreover, from the definition of variance, $\mathrm{E}[S_i^2] = \mathrm{Var}[S_i] + \frac{1}{n^2}$. Equation (6) then can be written in the form

$$x^* = \frac{r}{\bar{k}} \left( \frac{n-1}{n^2} - \mathrm{Var}[S_i] \right). \tag{7}$$

Equation (7) has a very intuitive structure. First, in the degenerate case when $\mathrm{Var}[S_i] = 0$ it collapses into the well-known expression for the symmetric equilibrium investment in a group contest where all group sizes are fixed and equal to $\bar{k}$,

$$x^0 = \frac{r(n-1)}{\bar{k} n^2}. \tag{8}$$

Second, when group sizes are stochastic, $\mathrm{Var}[S_i] \geq 0$. Moreover, the variance is strictly positive unless group sizes are perfectly correlated. Therefore, we obtain the following result.

**Proposition 1** *For symmetrically distributed group sizes, the symmetric equilibrium investment is not higher than in the case when the group size is fixed at $\bar{k}$; that is, $x^* \leq x^0$. Moreover,*

subsets of a Euclidean space; (ii) aggregator function $U(x, y) = \sum_{i,j} \pi_{ij}(x_{ij}, y_{-(ij)})$ is diagonally transfer continuous in $y$; and (iii) $U(x, y)$ is diagonally transfer quasiconcave in $x$. Here, $x = (x_{11}, \dots, x_{1m_1}, \dots, x_{n1}, \dots, x_{nm_1})$ is a generic vector of all players' investments, and $y_{-(ij)}$ is a generic vector of investments of all players other than $ij$. Condition (i) is satisfied trivially by restricting investment spaces to $[0, 1]$. Condition (ii) holds because, as shown by Baye, Tian and Zhou (1993), the aggregator in contests between individuals satisfies the diagonal transfer continuity condition; hence aggregator $U$, which is a composition of individual payoffs with continuous functions, satisfies it as well. Finally, payoffs (3) are strictly concave in $x_{ij}$, therefore $U$ is strictly concave in $x$ (which is a stronger condition than diagonal transfer quasiconcavity).

*the inequality is strict unless (i) $p_{\boldsymbol{k}}$ is degenerate or (ii) $p_{\boldsymbol{k}} = 0$ for all $\boldsymbol{k} \neq \alpha\mathbf{1}$, for some $\alpha \in \{1, \ldots, m\}$.*

It also follows from Eq. (7) that the reduction in equilibrium investment due to uncertainty is stronger the larger the variance in relative group impacts.

We next explore the role of dependence between group sizes. One way to model such dependence is to assume that the size of group $i$ is given by $K_i = g(Z, Y_i)$, where $Z$ is an integer random variable common for all groups, $Y_i$ are i.i.d. integer idiosyncratic shocks independent of $Z$, and $g$ is an integer-valued function increasing in both arguments. One simple example is an additive model, $g(z, y_i) = z + y_i$, where the common and idiosyncratic components are perfect substitutes. Intuitively, as the size of the common component $Z$ increases, variation in relative group impacts should go down because, for any given realization of $Z$, group sizes become more similar. This intuition may not work, however, when there is strong complementarity between $Z$ and $Y_i$, because an increase in $Z$ can lead to an increase in the effect of $Y_i$ on $K_i$ and hence to a larger variation in relative impacts. The result is summarized in the following proposition.

**Proposition 2** *Suppose $\frac{g(z+1,y)}{g(z,y)}$ is decreasing (increasing) in $y$. Then a stochastic increase in $Z$ leads to a lower (higher)* $\mathrm{Var}[S_i]$.

Returning to the example discussed above, for $g(z, y_i) = z + y_i$ we have $\frac{g(z+1,y)}{g(z,y)} = \frac{z+1+y}{z+y}$ decreasing in $y$ and hence Proposition 2 indeed implies that a stochastic increase in $Z$ will lead to a reduction in $\mathrm{Var}[S_i]$. In fact, this example is a special case of a more general property.

**Corollary 1** *Suppose $g(z, y_i) = \phi(a(z) + b(y_i))$ where $a(\cdot)$ and $b(\cdot)$ are increasing and $\phi(\cdot)$ is increasing and log-concave (log-convex). Then a stochastic increase in $Z$ leads to a lower (higher)* $\mathrm{Var}[S_i]$.

Corollary 1 provides a straightforward way to construct examples where a stochastic increase in $Z$ leads to an increase in $\mathrm{Var}[S_i]$. For example, function $\phi(t) = 2^{t^2}$ is log-convex, and hence $g(z, y_i) = 2^{(z+y_i)^2}$ produces the desired result. Another example (not covered by Corollary 1, but easily verified via Proposition 2) is $g(z, y_i) = z^{y_i}$. As expected, in both cases, the common and idiosyncratic components are strong complements.

Next, we consider expected total investment in the contest, $\mathrm{E}[X^*] = n\bar{k}x^*$. Equation (7) gives

$$\mathrm{E}[X^*] = r\left(\frac{n-1}{n} - n\mathrm{Var}[S_i]\right). \tag{9}$$

Note that, for individual investment $x^*$, a stochastic increase in $Z$ has two effects. On the one hand, according to Proposition 2, when $\frac{g(z+1,y)}{g(z,y)}$ is decreasing in $y$, it leads to a lower $\mathrm{Var}[S_i]$, thereby increasing $x^*$. On the other hand, a stochastic increase in $Z$ also increases $\bar{k}$, thereby reducing $x^*$. In contrast, expected total investment in the contest is independent of $\bar{k}$. Thus, a stochastic increase in $Z$ only affects expected total investment in the contest through its effect on the variance of relative group impacts.

**Corollary 2** *(i) Expected total investment in the contest is decreasing in* $\mathrm{Var}[S_i]$.
*(ii) Suppose* $K_i = g(Z, Y_i)$ *as in Proposition 2 and* $\frac{g(z+1,y)}{g(z,y)}$ *is decreasing (increasing) in* $y$*. Then a stochastic increase in* $Z$ *leads to a higher (lower) expected total investment in the contest.*

In contests with population uncertainty, the contest designer may be able to disclose the number of participants; it is, therefore, of interest to explore whether commitment to such disclosure is optimal. Parallel results have been established in the literature on contests between individuals. Lim and Matros (2009) showed that disclosure does not affect *ex ante* expected aggregate investment in Tullock contests with the binomial distribution of the number of players. Fu, Jiao and Lu (2011) extended this result to contests with a CSF of the generalized lottery form and showed that disclosure can increase or decrease aggregate investment depending on the shape of the CSF's impact function. Ryvkin and Drugov (2017) further generalized these results to arbitrary tournaments with arbitrary distributions of the number of players.

In our setting, disclosing the number of players in each group will generate the same total equilibrium group investment $X_i^0 = k_i x_i^0 = \frac{r(n-1)}{n^2}$ in all groups, where $x_i^0 = \frac{r(n-1)}{k_i n^2}$ is the semi-symmetric equilibrium effort level in the corresponding group contest with commonly known group sizes $\boldsymbol{k}$ (Baik, 1993). The resulting aggregate contest investment, $X^0 = \frac{r(n-1)}{n}$,[7] exceeds the expected total investment without disclosure, Eq. (9), in all but degenerate cases.

**Corollary 3** *The disclosure of group sizes leads to an increase in expected total investment. The effect is strictly positive with the exception of the degenerate cases in Proposition 1.*

Next, we provide an example to illustrate the results presented in this section, using a symmetric multivariate distribution with dependence between group sizes.

**Example 1: Symmetric distribution.** Consider the additive model described above, with $K_i = Z + Y_i$. Further, assume that $Z$ is a Poisson random variable with parameter $\theta$, and $Y_i$, $i = 1, \dots, n$, are i.i.d. zero-truncated Poisson random variables with parameter $\lambda$. The pmf for the zero-truncated Poisson distribution is given by

$$\Pr[Y_i = y | Y_i \geq 1] = \frac{\lambda^y}{y!(e^\lambda - 1)}. \tag{10}$$

The joint pmf of $\boldsymbol{K}$ is, therefore,[8]

$$p_{\boldsymbol{k}} = \frac{e^{-\theta} \lambda^{\sum_{i=1}^n k_i}}{(e^\lambda - 1)^n \prod_{i=1}^n k_i!} \sum_{s=0}^{\min\{k_1, \dots, k_n\} - 1} (s!)^{n-1} \left(\frac{\theta}{\lambda^n}\right)^s \prod_{i=1}^n \binom{k_i}{s}, \quad \boldsymbol{k} \geq \mathbf{1}. \tag{11}$$

[7] Aggregate investment $X^0$ in a contest where group sizes are disclosed is the same in any equilibrium, not just in the semi-symmetric one.

[8] This joint pmf can be derived using the same approach as to deriving the standard multivariate Poisson distribution in which the $Y_i$, $i = 1, \dots, n$ are not truncated at zero, (cf., e.g., Johnson, Kotz and Balakrishnan, 1997).

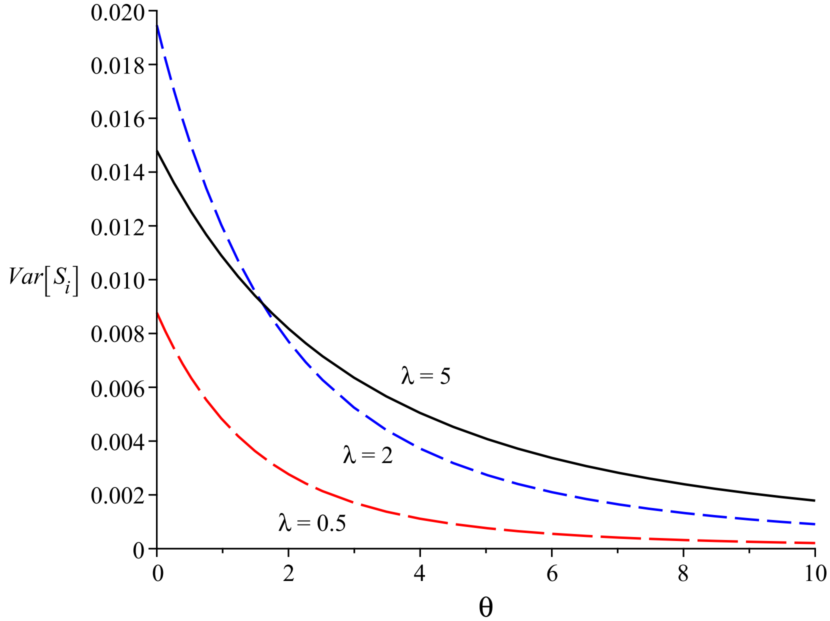


Figure 1: Variance of $S_i$ for $\lambda \in \{0.5, 2, 5\}$ and $\theta$ varying from 0 to 10.

Since $g(Z, Y_i)$ is additive, $\frac{g(z+1,y)}{g(z,y)}$ is decreasing in $y$. Thus, by Proposition 2, a stochastic increase in $Z$ will lead to a reduction in $\text{Var}[S_i]$. For the Poisson distribution, an increase in $\theta$ generates a stochastic increase in $Z$.

To illustrate the main results from this section, we consider the case of $n = 3$ groups. For simplicity, we set $r = 1$. We compute $\text{Var}[S_i]$ directly using (11) and the definition of $S_i$. Figure 1 plots $\text{Var}[S_i]$ as a function of $\theta$ for three different values of $\lambda$. For each case, $\lambda \in \{0.5, 2, 5\}$, the variance of relative group impact is strictly decreasing as $\theta$ increases. Furthermore, as $\theta \to \infty$, $\text{Var}[S_i] \to 0$.

The effects of $\theta$ and $\text{Var}[S_i]$ on equilibrium individual investment are also highlighted in Figure 2, which plots the equilibrium investment under group size uncertainty alongside the corresponding equilibrium investment in a deterministic contest with all group sizes fixed and equal to $\bar{k}$. For the multivariate distribution (11) with $n = 3$, the mean group size as a function of $\theta$ and $\lambda$ is given by

$$\bar{k}(\theta, \lambda) = \theta + \frac{\lambda e^{\lambda}}{e^{\lambda} - 1}.$$

To make clear the relevant comparison, we denote by $x^0(\theta, \lambda)$ the equilibrium investment in the corresponding deterministic contest with $\bar{k}(\theta, \lambda)$ active participants in each group. As expected, $x^*(\theta, \lambda) < x^0(\theta, \lambda)$, for each $(\theta, \lambda)$. However, keeping $\lambda$ fixed, as $\theta$ increases, the difference between the equilibrium investment with and without group size uncertainty disappears.

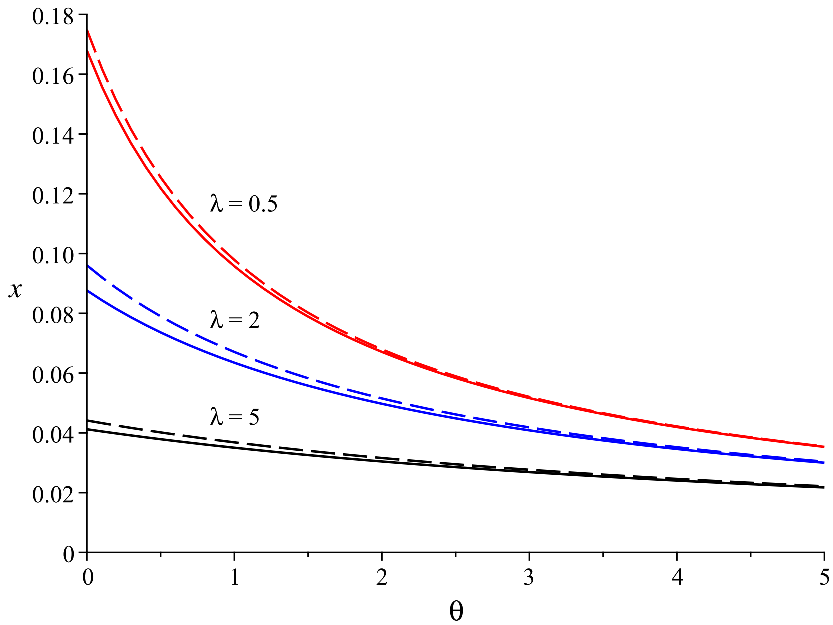


Figure 2: Equilibrium individual investment, $x^*(\theta, \lambda)$ (solid lines), compared with $x^0(\theta, \lambda)$ (dashed lines), the corresponding equilibrium investment in a group contest where all group sizes are fixed and equal to $\bar{k}(\theta, \lambda)$.

## 4 Arbitrary group size distributions with $n = 2$

In this section, we consider the case where there are only $n = 2$ groups, but the joint distribution over group sizes need not be symmetric. The first-order conditions (5) take the form

$$\sum_{\boldsymbol{k}} p_{\boldsymbol{k}} \frac{k_1^r (x_1^*)^{r-1} (k_2 x_2^*)^r}{((k_1 x_1^*)^r + (k_2 x_2^*)^r)^2} = \frac{\bar{k}_1}{r}, \quad \sum_{\boldsymbol{k}} p_{\boldsymbol{k}} \frac{k_2^r (x_2^*)^{r-1} (k_1 x_1^*)^r}{((k_1 x_1^*)^r + (k_2 x_2^*)^r)^2} = \frac{\bar{k}_2}{r}, \tag{12}$$

which immediately implies

$$\frac{x_2^*}{x_1^*} = \frac{\bar{k}_1}{\bar{k}_2}. \tag{13}$$

That is, the ratio of equilibrium investment levels for the individual members of group 2 relative to group 1 is equal to the inverse ratio of the expected number of players in group 2 relative to group 1. Furthermore, this implies that the *expected* equilibrium group level investment, $\mathrm{E}[X_i^*] = \bar{k}_i x_i^*$, is identical across groups.

Using (13) and (12), we obtain

$$x_i^* = \frac{r}{\bar{k}_i} \sum_{\boldsymbol{k}} p_{\boldsymbol{k}} \frac{(k_1 k_2 \bar{k}_1 \bar{k}_2)^r}{(k_1^r \bar{k}_2^r + \bar{k}_1^r k_2^r)^2}. \tag{14}$$

First, notice that for a group contest with deterministic group sizes equal to $\bar{k}_1$ and $\bar{k}_2$, equation

(14) reduces to

$$x_i^0 = \frac{r}{4\bar{k}_i}, \tag{15}$$

which corresponds to the equilibrium derived in Baik (1993, 2008). Second, it follows that in equilibrium, total expected investment in the contest, $\mathrm{E}[X^*] = \mathrm{E}[X_1^* + X_2^*] = 2\mathrm{E}[X_1^*]$, is given by

$$\mathrm{E}[X^*] = 2r \sum_{\boldsymbol{k}} p_{\boldsymbol{k}} \frac{(k_1 k_2 \bar{k}_1 \bar{k}_2)^r}{(k_1^r \bar{k}_2^r + \bar{k}_1^r k_2^r)^2}. \tag{16}$$

As seen from (16), $\mathrm{E}[X^*] = \frac{r}{2}$ if $K_1 = aK_2$ for some $a > 0$ or if both $K_1$ and $K_2$ are degenerate (even if they are different). However, in all other non-deterministic cases, expected total investment is lower.

**Proposition 3** *For $n = 2$, with an arbitrary distribution of group sizes, expected total investment is lower as compared to the case when the group sizes are fixed at $(\bar{k}_1, \bar{k}_2)$; that is, $\mathrm{E}[X^*] \leq \frac{r}{2}$. The inequality is strict unless $p_{\boldsymbol{k}}$ is degenerate or $K_1 = aK_2$ for some $a > 0$.*

Similar to the case of symmetric group size distributions (cf. Corollary 3), Proposition 3 informs on the consequences of disclosure of group sizes $(k_1, k_2)$. Total equilibrium investment with disclosure, $X^0 = \frac{r}{2}$, exceeds expected total investment without disclosure in all but the degenerate cases.

**Corollary 4** *For $n = 2$, with an arbitrary distribution of group sizes, the disclosure of group sizes leads to an increase in expected total investment. The effect is strictly positive with the exception of the degenerate cases in Proposition 3.*

We illustrate these results using two examples. Example 2 uses an asymmetric distribution constructed using Poisson random variables (following an approach similar to the one used for Example 1). This example illustrates the effect of positive correlation between group sizes. Example 3 considers the case in which there is always a fixed total number of active players $m$, divided between the two groups according to a Binomial distribution, in order to illustrate the effect of negative correlation between group sizes.

**Example 2: Asymmetric distribution with positive correlation.** Similar to the construction in Example 1, let $\boldsymbol{K} = (K_1, K_2)$ be given by $K_i = Z + Y_i$, where $Z$ is a Poisson random variable with parameter $\theta$, and $Y_i$, $i = 1, 2$, are independent zero-truncated Poisson random variables with (possibly different) parameters $\lambda_i$. The joint pmf of $\boldsymbol{K}$ is given by

$$p_{\boldsymbol{k}} = \frac{e^{-\theta} \lambda_1^{k_1} \lambda_2^{k_2}}{(e^{\lambda_1} - 1)(e^{\lambda_2} - 1) k_1! k_2!} \sum_{s=0}^{\min\{k_1, k_2\} - 1} \binom{k_1}{s} \binom{k_2}{s} s! \left( \frac{\theta}{\lambda_1 \lambda_2} \right)^s, \quad \boldsymbol{k} \geq \mathbf{1}. \tag{17}$$

Note that the mean number of active players in group $i = 1, 2$ as a function of $\theta$ and $\lambda_i$ is

$$\bar{k}_i(\theta, \lambda_i) = \theta + \frac{\lambda_i e^{\lambda_i}}{e^{\lambda_i} - 1}. \tag{18}$$

Again, for simplicity, we set $r = 1$. Then, using equation (14), we compute $\mathrm{E}[X^*]$ for different combinations of parameters $(\theta, \lambda_1, \lambda_2)$.

For any fixed pair of parameters $(\lambda_1, \lambda_2)$, as $\theta$ increases, $Z$ becomes more important and the idiosyncratic components, $Y_i$, $i = 1, 2$, become less important for the realized group size. Consequently, realizations of $\boldsymbol{K}$ with $K_1 = K_2$ will become relatively more likely. For these realizations of $\boldsymbol{K}$, the term in the summand of equation (14) is equal to $\frac{p_{\boldsymbol{k}}}{4}$. Thus, intuition suggests that as $\theta$ increases, these terms receive greater probability weight, and $\mathrm{E}[X^*]$ will tend to increase. Although this argument seems intuitive, it does not always hold, as we show in the examples below.

In Figure 3, we plot $\mathrm{E}[X^*]$ for $(\lambda_1, \lambda_2) = (2, 5)$ and for $(\lambda_1, \lambda_2) = (8, 5)$, with $\theta$ varying from 0 to 10. In both cases, expected total investment is below the corresponding total investment in a contest with deterministic group sizes, which is represented by the horizontal reference line at 0.5. Furthermore, as $\theta$ increases, expected total investment is monotonically increasing, consistent with the preceding intuition.

In contrast, Figure 4 provides an example where, if the asymmetry between groups is strong enough and one of the groups has a sufficiently low parameter $\lambda_i$, expected total investment may not be monotonically increasing in $\theta$. Using $(\lambda_1, \lambda_2) = (1, 15)$ and $(\lambda_1, \lambda_2) = (0.5, 15)$, Figure 4 shows that $\mathrm{E}[X^*]$ is, at first, decreasing in $\theta$, then subsequently increasing in $\theta$. The source of this nonmonotonicity in $\mathrm{E}[X^*]$ with respect to $\theta$ is the fact that we use zero-truncated Poisson random variables for the idiosyncratic components, $Y_i$. Specifically, if $\lambda_2$ is relatively large, while $\lambda_1$ is sufficiently small, the truncated distribution for $Y_1$ is substantially different from its standard Poisson distribution, while the truncated distribution for $Y_2$ is very similar to its standard Poisson distribution. This differential impact of truncation on the distributions of $Y_1$ and $Y_2$ then distorts the relative likelihood of realizations in which group sizes are the same, provided $\theta$ is also sufficiently small. Nevertheless, even in these somewhat unusual cases, once $\theta$ grows sufficiently large, $\mathrm{E}[X^*]$ is increasing in $\theta$, as can be observed in Figure 4.

**Example 3: Negative correlation between group sizes.** In this example, we consider the effects of negative correlation between group sizes on the equilibrium investment in the contest. Suppose there are $m$ potential participants in the population. Each potential participant is active in one of the two groups. Let $q \in [0, 1]$ be the probability of any given player being a member of group 1. In this setting, $K_1 + K_2 = m$, from which the perfect negative correlation between group sizes is evident. Then the probability of $\boldsymbol{K} = \boldsymbol{k} = (k_1, m - k_1)$ is given by the binomial probability

$$p^B_{\boldsymbol{k}} = \binom{m}{k_1} q^{k_1} (1-q)^{m-k_1}, \quad k_1 = 0, \ldots, m. \tag{19}$$

Once again, since we assume the minimum group size in any group is 1, we use a truncated distribution, updated to ensure that $k_1 \geq 1$ and $k_1 \leq m - 1$ (corresponding to $k_2 \geq 1$). The

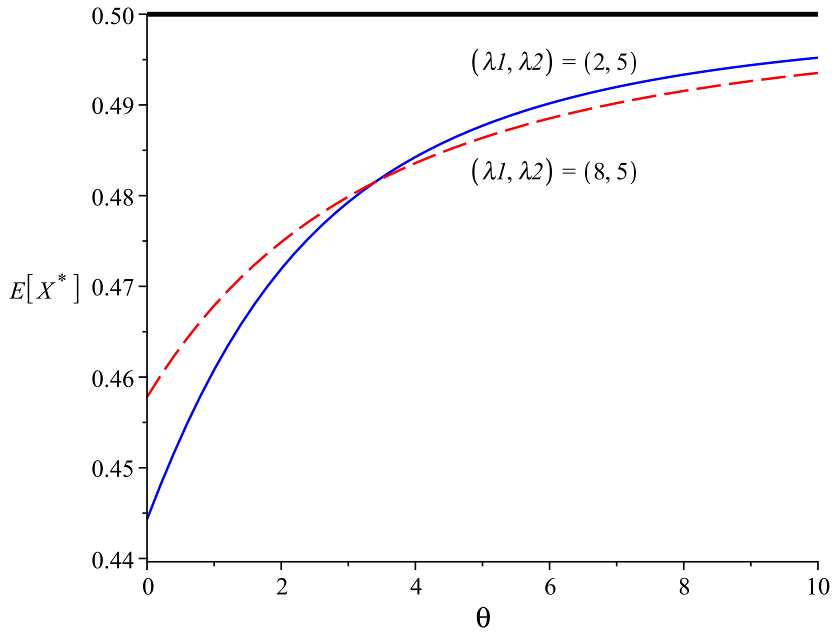


Figure 3: Expected total investment $\mathrm{E}[X^*]$ for $(\lambda_1, \lambda_2) = (2, 5)$ and $(\lambda_1, \lambda_2) = (8, 5)$.

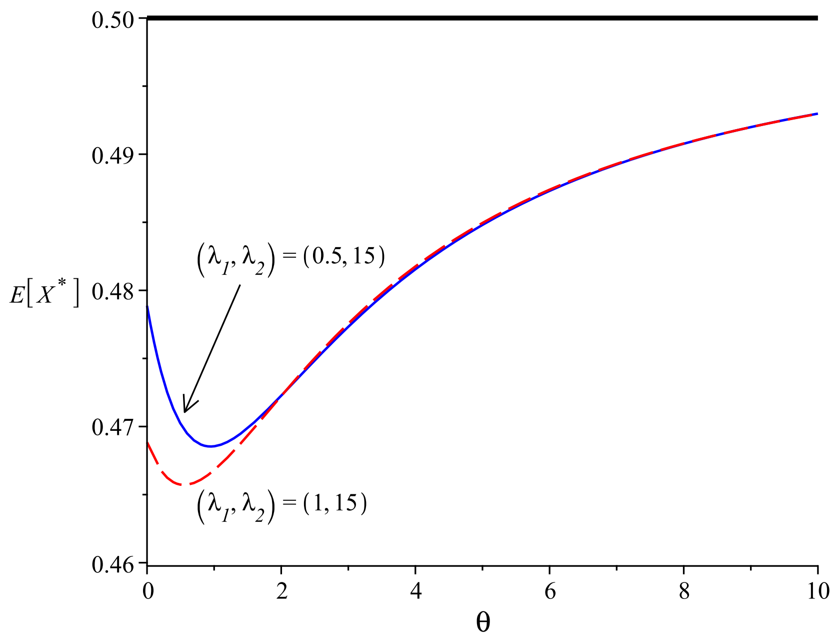


Figure 4: Expected total investment $\mathrm{E}[X^*]$ for $(\lambda_1, \lambda_2) = (0.5, 15)$ and $(\lambda_1, \lambda_2) = (1, 15)$.

resulting pmf is given by

$$p_{\boldsymbol{k}} = \frac{\binom{m}{k_1} q^{k_1} (1-q)^{m-k_1}}{1 - (q^m + (1-q)^m)}, \quad k_1 = 1, \ldots, m-1. \tag{20}$$

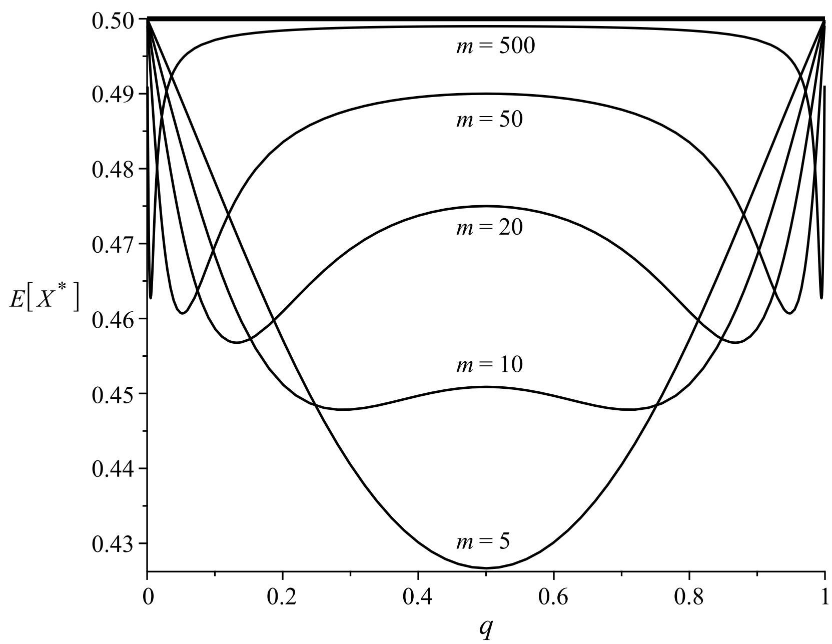


Figure 5: Expected total investment $\mathrm{E}[X^*]$ as a function of $q$, for $m \in \{5, 10, 20, 50, 500\}$. The reference line at 0.50 indicates equilibrium total investment in a contest without group size uncertainty.

The resulting mean group sizes are given by

$$\bar{k}_1 = mq \left( \frac{1 - q^{m-1}}{1 - (q^m + (1-q)^m)} \right), \quad \bar{k}_2 = m(1-q) \left( \frac{1 - (1-q)^{m-1}}{1 - (q^m + (1-q)^m)} \right).$$

Fixing $r = 1$ and using equation (14), we compute $\mathrm{E}[X^*]$ for various parameters $(m, q)$. In Figure 5 we plot $\mathrm{E}[X^*]$ against $q$ for $m \in \{5, 10, 20, 50, 500\}$. Several features are worth highlighting. First, as predicted, expected total investment is below the corresponding total investment in a contest without group size uncertainty. Second, when $q$ is equal to 0.5 (so that each group is equally likely to receive a particular participant), expected total investment is increasing in the population size. For $q$ close to 0.5, this ordering is preserved. Intuitively, when $q$ is close to 0.5, even if there is perfect negative correlation, realized group sizes are much more likely to be the same, especially when the number of potential participants is very large.

However, when there are stronger asymmetries between groups, i.e., when $q$ is closer to 0 or closer to 1, expected total investment may be higher for smaller populations than for larger ones. For example, as shown in Figure 5, when $q = 0.1$ (or $q = 0.9$), $\mathrm{E}[X^*]$ is higher for $m = 5$ than for $m = 10$ or $m = 20$.

**The group size paradox revisited.** The *group size paradox*, discussed initially by Olson (1965), is a feature of group contests whereby a smaller group is more likely to win the contest because its members have more to gain from such a win and are less prone to free-riding on the investments of one another. Multiple authors have since addressed the emergence and robustness of this phenomenon in models of group contests, showing that it is not universal and can be reversed, for example, when the costs of investment are sufficiently convex (Esteban and Ray, 2001) or in the presence complementarities in group production (Kolmar and Rommeswinkel, 2011). In this section, we explore how group size affects the group's chances of winning in the presence of group size uncertainty. To keep things simple, we assume that there are two groups and the groups' sizes $(K_1, K_2)$ are independent.

Note first that when group sizes are deterministic ($(K_1, K_2) = (\bar{k}_1, \bar{k}_2)$ with probability one), we obtain a version of group size paradox whereby each group's total equilibrium investment, $X_i^* = \bar{k}_i x_i^0 = \frac{r}{4}$, and hence its probability of winning, $\frac{1}{2}$, is independent of group size, cf. Eq. (15).[9] When group sizes are stochastic, the equilibrium probability of group 1 winning is given by

$$w_1 = \sum_{\boldsymbol{k}} p_{\boldsymbol{k}} \frac{(k_1 x_1^*)^r}{(k_1 x_1^*)^r + (k_2 x_2^*)^r} = \sum_{\boldsymbol{k}} p_{\boldsymbol{k}} \frac{k_1^r \bar{k}_2^r}{k_1^r \bar{k}_2^r + k_2^r \bar{k}_1^r} = \mathrm{E} \frac{\tilde{K}_1^r}{\tilde{K}_1^r + \tilde{K}_2^r}, \tag{21}$$

where we used Eq. (13) and defined rescaled group sizes $\tilde{K}_i = \frac{K_i}{\bar{k}_i}$. Note that $\mathrm{E}[\tilde{K}_i] = 1$.

Consider an increase in the size of group 1 in the sense of first-order stochastic dominance (FOSD). The expectation of $\tilde{K}_1$ is not affected, and the function under expectation in Eq. (21) is strictly increasing and strictly concave in $\tilde{K}_1$. Therefore, an increase in $\tilde{K}_1$ in the sense of *second-order* stochastic dominance (SOSD) will lead to an increase in $w_1$. Thus, we have the following result.

**Lemma 3** *In a contest between two groups with independent stochastic sizes* $(K_1, K_2)$*, the equilibrium probability of group* $i$ *winning increases in* $K_i$ *(when* $K_i$ *is increased in the FOSD sense) if* $\tilde{K}_i$ *increases in the SOSD sense.*

As we show with the following example, when $K_i$ is stochastically increased, it is possible for $\tilde{K}_i$ to go up or down in the SOSD sense, depending on the details of its distribution. Suppose $K_1$ takes two positive integer values, $a$ and $b$, with probability $\frac{1}{2}$ each, such that $b > a \geq 1$. In this case $\bar{k}_1 = \frac{a+b}{2}$, and $\tilde{K}_1$ takes values $\frac{2a}{a+b}$ and $\frac{2b}{a+b}$ with probabilities $\frac{1}{2}$. This gives $\mathrm{Var}(\tilde{K}_1) = \frac{(b-a)^2}{(b+a)^2}$. An increase in $a$ and $b$ such that $b - a$ stays constant, therefore, leads to a reduction in $\mathrm{Var}(\tilde{K}_1)$, whereas an increase in $b$, keeping $a$ constant, leads to an increase in $\mathrm{Var}(\tilde{K}_1)$.

We conclude that, when group sizes are stochastic, an FOSD increase in group size $K_i$ has no universal effect on the group's winning probability – it can go up or down depending on the riskiness of rescaled group size $\tilde{K}_i$, which can change in either direction. Thus, the presence of uncertainty in group sizes is another dimension that potentially reduces the robustness of the group size paradox. A more detailed exploration in this area can be of interest.

[9] This is a generic property of models with perfect substitutes and linear investment costs where the group public good is non-rival (Baik, 1993). If the public good were rival, i.e., if the prize were divided equally among the members of the winning group, as in the original formulation of Olson (1965), the probability of winning would be decreasing in group size.

## 5 Concluding remarks

Altogether, our results suggest that population uncertainty has a negative effect on equilibrium investment in group contests. In addition, we show that, depending on the nature and the extent of the correlations between group sizes, this effect may be magnified or moderated when there is a common shock, such as a change in regulatory policy, the approach of an election, or a landmark judicial decision. Our study also motivates some promising avenues for future work. For example, while our analysis assumes group sizes are stochastic, the true source of such group size uncertainty may be the endogenous entry decisions of potential participants with commonly aligned interests. Any investigation of endogenous entry into group contests, or the formation of groups and alliances in contests, has the potential to generate population uncertainty for participants at the time of investment.

The structure of group size uncertainty we considered is rather stylized in that all players *ex ante* have the same information. A more general, and nuanced, model would allow for informational asymmetries, with insiders better informed about the group size than outsiders. The extreme version of such a setting where insiders know their group size with certainty leads to the same aggregate outcome as the contest with complete information. However, a setting where insiders only have partial information, e.g., they receive signals about their group's size, would be an interesting extension.

Other natural extensions would be to consider a group contest in which the prize is partially or fully rival, or in which the individual investments of group members are aggregated according to a different production technology, such as weak-link, best-shot, or with intermediate levels of complementarity.

Finally, while our results inform on the consequence of (non)disclosure of the sizes of competing groups by the contest designer, our study raises additional questions about whether or when it may be optimal for participants to conceal or reveal their own participation in a group contest. For instance, are organizations that conceal their lobbying efforts from the public eye acting optimally, or could they improve their expected payoff by publicly declaring their support (or intended support) to encourage increased participation? Our investigation provides an initial foundation for future analysis of these kinds of questions.

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# A Proofs

**Proof of Lemma 1** Without loss, consider players 11 and 12 in group 1. Let $Y_{-1} = \sum_{h=2}^{n} X_h^r$ denote the aggregate output of groups other than group 1 for a given realization of other groups' sizes $\boldsymbol{k}_{-1}$. Note that, for a given $\boldsymbol{k}_{-1}$, $Y_{-1}$ is still random because different combinations of players can be drawn in each group, potentially leading to different group investments $X_h$ if individual investments within groups are not the same. We will use $\mathrm{E}_{-1}$ to denote expectation over all such draws, for a given $\boldsymbol{k}_{-1}$. Further, we will use $\mathrm{E}_{k_1}^{1j}$ to denote expectations over possible draws of $k_1$ players in group 1 that include player $1j$, which we will write out explicitly.

Using this notation, Eq. (3) for the payoff of player 11 becomes

$$\pi_{11} = \frac{1}{\bar{k}_1} \sum_{\boldsymbol{k}_{-1}} \Bigg[ p_{1\boldsymbol{k}_{-1}} \mathrm{E}_{-1} \frac{x_{11}^r}{x_{11}^r + Y_{-1}} + 2 p_{2\boldsymbol{k}_{-1}} \mathrm{E}_{-1} \mathrm{E}_2^{11} \frac{(x_{11} + x_{1j})^r}{(x_{11} + x_{1j})^r + Y_{-1}}$$
$$+ 3 p_{3\boldsymbol{k}_{-1}} \mathrm{E}_{-1} \mathrm{E}_3^{11} \frac{(x_{11} + x_{1j} + x_{1l})^r}{(x_{11} + x_{1j} + x_{1l})^r + Y_{-1}} + \dots \Bigg] - x_{11}. \quad (22)$$

Here,

$$\mathrm{E}_2^{11}\frac{(x_{11}+x_{1j})^r}{(x_{11}+x_{1j})^r+Y_{-1}} = \frac{1}{m_1-1}\sum_{2\le j\le m_1}\frac{(x_{11}+x_{1j})^r}{(x_{11}+x_{1j})^r+Y_{-1}}, \tag{23}$$

$$\mathrm{E}_3^{11}\frac{(x_{11}+x_{1j}+x_{1l})^r}{(x_{11}+x_{1j}+x_{1l})^r+Y_{-1}} = \frac{2}{(m_1-1)(m_1-2)}\sum_{2\le j,l\le m_1, j\neq l}\frac{(x_{11}+x_{1j}+x_{1l})^r}{(x_{11}+x_{1j}+x_{1l})^r+Y_{-1}},$$

etc. That is, for each realization of $k_1$, expectation $\mathrm{E}_{k_1}^{11}$ is the average over all equally likely draws of $k_1$ players from group 1 that include player 11.

Payoff function (22) is strictly concave in $x_{11}$; therefore, $\frac{\partial \pi_{11}}{\partial x_{11}} \le 0$ in any (pure) equilibrium, with equality for $x_{11} > 0$. Suppose first that $x_{11}, x_{12} > 0$. Equation (22) and a similar equation for $\pi_{12}$, the payoff of player 12, then give

$$\frac{r}{\bar{k}_1}\sum_{\boldsymbol{k}_{-1}}\Bigg[p_{1\boldsymbol{k}_{-1}}\mathrm{E}_{-1}\frac{x_{11}^{r-1}Y_{-1}}{(x_{11}^r+Y_{-1})^2} + 2p_{2\boldsymbol{k}_{-1}}\mathrm{E}_{-1}\mathrm{E}_2^{11}\frac{(x_{11}+x_{1j})^{r-1}Y_{-1}}{[(x_{11}+x_{1j})^r+Y_{-1}]^2} \tag{24}$$

$$+3p_{3\boldsymbol{k}_{-1}}\mathrm{E}_{-1}\mathrm{E}_3^{11}\frac{(x_{11}+x_{1j}+x_{1l})^{r-1}Y_{-1}}{[(x_{11}+x_{1j}+x_{1l})^r+Y_{-1}]^2}+\ldots\Bigg] = 1,$$

$$\frac{r}{\bar{k}_1}\sum_{\boldsymbol{k}_{-1}}\Bigg[p_{1\boldsymbol{k}_{-1}}\mathrm{E}_{-1}\frac{x_{12}^{r-1}Y_{-1}}{(x_{12}^r+Y_{-1})^2} + 2p_{2\boldsymbol{k}_{-1}}\mathrm{E}_{-1}\mathrm{E}_2^{12}\frac{(x_{12}+x_{1j})^{r-1}Y_{-1}}{[(x_{12}+x_{1j})^r+Y_{-1}]^2}$$

$$+3p_{3\boldsymbol{k}_{-1}}\mathrm{E}_{-1}\mathrm{E}_3^{12}\frac{(x_{12}+x_{1j}+x_{1l})^{r-1}Y_{-1}}{[(x_{12}+x_{1j}+x_{1l})^r+Y_{-1}]^2}+\ldots\Bigg] = 1.$$

Here, expectations $\mathrm{E}_{k_1}^{12}$ are defined similar to (23).

Next, we set the left-hand sides of first-order conditions (24) equal to each other. Note that expectation $\mathrm{E}_2^{11}$ and $\mathrm{E}_2^{12}$ both contain a term with the sum $(x_{11}+x_{12})$, which will cancel out. Similarly, expectations $\mathrm{E}_3^{11}$ and $\mathrm{E}_3^{12}$ both contain terms with the sum $(x_{11}+x_{12}+x_{1l})$ for all $l > 2$, which will cancel out as well. Generally, terms of the form $(x_{11}+x_{12}+x_{1l_3}+\ldots+x_{1l_{k_1}})$ will cancel each other out in $\mathrm{E}_{k_1}^{11}$ and $\mathrm{E}_{k_1}^{12}$ for each $k_1 = 2,\ldots,m_1-1$. Finally, the terms with $k_1 = m_1$ will cancel out completely.

The remaining terms in $\mathrm{E}_{k_1}^{11}$ will only contain sums $(x_{11}+x_{1l_2}+\ldots+x_{1l_{k_1}})$ for $l_2,\ldots,l_{k_1} > 2$, and similarly the remaining terms in $\mathrm{E}_{k_1}^{12}$ will only contain sums $(x_{12}+x_{1l_2}+\ldots+x_{1l_{k_1}})$ for $l_2,\ldots,l_{k_1} > 2$. Thus, as long as $K_1$ is non-degenerate, i.e., it is not equal $m_1$ with certainty, we have an equality of the form $G(x_{11},\cdot) = G(x_{12},\cdot)$, where $G$ is strictly decreasing in the first argument. This implies $x_{11} = x_{12}$.

Suppose now that $x_{11} = 0$ and $x_{12} > 0$. Then the first of the first-order conditions (24) holds with inequality $\le$, while the second still holds with equality. This implies the left-hand side of the first condition is less than the left-hand side of the second one, i.e., using the notation above, $G(0,\cdot) \le G(x_{12},\cdot)$. But this is impossible because $G$ is strictly decreasing in the first argument. Finally, if $x_{11} = x_{12} = 0$, we have the result as well. ■

**Proof of Lemma 2** It is sufficient to show that the payoff function $\pi_{ij}(x_{ij},(x_h^*)_{h=1}^n)$ in Eq. (4)

is globally strictly concave in $x_{ij}$. This follows immediately from the observation that function $\frac{x}{x+a}$ is strictly concave in $x$ for any $a > 0$, function $(x+b)^r$ is concave in $x$ for any $b \geq 0$ and $r \in (0,1]$, and a composition of a strictly concave and concave functions is strictly concave. It then follows that if $(x_i^*)_{i=1}^n$ is an interior solution to the system of $n$ equations in (5), setting $x_{ij} = x_i^*$ is a best response for player $ij$, and hence $(x_i^*)_{i=1}^n$ is a Nash equilibrium. ■

**Proof of Proposition 2** Since $\mathrm{Var}[S_i] = \mathrm{E}[S_i^2] + \frac{1}{n^2}$, we will show that $\mathrm{E}[S_i^2]$ behaves as stated in the proposition. By the law of iterated expectations, $\mathrm{E}[S_i^2] = \mathrm{E}_Z[\mathrm{E}_{\boldsymbol{Y}}[S_i^2|Z]]$, where the inner expectation is taken over the realizations of $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ for a given $Z$ and the outer expectation is with respect to $Z$. In order to prove the result stated in the proposition, it is sufficient to establish that $\mathrm{E}_{\boldsymbol{Y}}[S_i^2|Z = z]$ is decreasing (increasing) in $z$ when $\frac{g(z+1,y)}{g(z,y)}$ is decreasing (increasing). Using the definition of $S_i$ and symmetry, we have

$$\mathrm{E}_{\boldsymbol{Y}}[S_i^2|Z = z+1] - \mathrm{E}_{\boldsymbol{Y}}[S_i^2|Z = z] = \frac{1}{n}\sum_i \left(\mathrm{E}_{\boldsymbol{Y}}[S_i^2|Z = z+1] - \mathrm{E}_{\boldsymbol{Y}}[S_i^2|Z = z]\right)$$
$$= \frac{1}{n}\mathrm{E}_{\boldsymbol{Y}} \sum_i \left( \frac{g(z+1,Y_i)^2}{(\sum_j g(z+1,Y_j))^2} - \frac{g(z,Y_i)^2}{(\sum_j g(z,Y_j))^2} \right) = \frac{1}{n}\mathrm{E}_{\boldsymbol{Y}} A_1(z,\boldsymbol{Y}) + \frac{1}{n}\mathrm{E}_{\boldsymbol{Y}} A_2(z,\boldsymbol{Y}),$$

where

$$A_1(z,\boldsymbol{y}) = \sum_i \frac{g(z+1,y_i)}{\sum_j g(z+1,y_j)} \left( \frac{g(z+1,y_i)}{\sum_j g(z+1,y_j)} - \frac{g(z,y_i)}{\sum_j g(z,y_j)} \right),$$
$$A_2(z,\boldsymbol{y}) = \sum_i \frac{g(z,y_i)}{\sum_j g(z,y_j)} \left( \frac{g(z+1,y_i)}{\sum_j g(z+1,y_j)} - \frac{g(z,y_i)}{\sum_j g(z,y_j)} \right).$$

We will establish that $A_1(z,\boldsymbol{y}) \leq (\geq) 0$ and $A_2(z,\boldsymbol{y}) \leq (\geq) 0$ provided $\frac{g(z+1,y)}{g(z,y)}$ is decreasing (increasing) in $y$. The sign of $A_1(z,\boldsymbol{y})$ is determined by the sign of the expression

$$B_1(z,\boldsymbol{y}) = \sum_i g(z+1,y_i) \left[ g(z+1,y_i) \sum_j g(z,y_j) - g(z,y_i) \sum_j g(z+1,y_j) \right]$$
$$= \sum_{i,j} g(z+1,y_i) g(z,y_i) g(z,y_j) \left[ \frac{g(z+1,y_i)}{g(z,y_i)} - \frac{g(z+1,y_j)}{g(z,y_j)} \right]$$

Suppose, for concreteness, that $\frac{g(z+1,y)}{g(z,y)}$ is decreasing in $y$. Then

$$\sum_{i,j} [g(z+1,y_i) - g(z+1,y_j)] g(z,y_i) g(z,y_j) \left[ \frac{g(z+1,y_i)}{g(z,y_i)} - \frac{g(z+1,y_j)}{g(z,y_j)} \right] \leq 0,$$

because the two expressions in square brackets have opposite signs. This gives

$$
\begin{aligned}
B_1(z,\boldsymbol{y}) &= \sum_{i,j} g(z+1,y_i)g(z,y_i)g(z,y_j)\left[\frac{g(z+1,y_i)}{g(z,y_i)} - \frac{g(z+1,y_j)}{g(z,y_j)}\right] \\
&\le \sum_{i,j} g(z+1,y_j)g(z,y_i)g(z,y_j)\left[\frac{g(z+1,y_i)}{g(z,y_i)} - \frac{g(z+1,y_j)}{g(z,y_j)}\right] \\
&= \sum_{j,i} g(z+1,y_i)g(z,y_j)g(z,y_i)\left[\frac{g(z+1,y_j)}{g(z,y_j)} - \frac{g(z+1,y_i)}{g(z,y_i)}\right] = -B_1(z,\boldsymbol{y}),
\end{aligned}
$$

which implies $B_1(z,\boldsymbol{y}) \le 0$. The derivation for $A_2(z,\boldsymbol{y})$ and the case when $\frac{g(z+1,y)}{g(z,y)}$ is increasing in $y$ is similar. ■

**Proof of Corollary 1** Suppose, for concreteness, that $\phi$ is log-concave. This implies that $\frac{\phi(t+x)}{\phi(x)}$ is decreasing in $x$ for any $t \ge 0$. Letting $x = x_2 + y$ and $t = x_1 - x_2 + y$, where $x_1 \ge x_2$, it follows that $\frac{\phi(x_1+y)}{\phi(x_2+y)}$ is decreasing in $y$ for any $x_1 \ge x_2$. Therefore, for any $y_1 \ge y_2$ we have $\frac{\phi(x_1+y_1)}{\phi(x_2+y_1)} \le \frac{\phi(x_1+y_2)}{\phi(x_2+y_2)}$. Setting $x_1 = a(z+1)$, $x_2 = a(z)$, $y_1 = b(y+1)$ and $y_2 = b(y)$, obtain that $\frac{g(z+1,y)}{g(z,y)}$ is decreasing in $y$. The case of log-convex $\phi$ is similar. ■

**Proof of Proposition 3** From equation (14),

$$
\mathrm{E}[X_1^*] = \mathrm{E}[X_2^*] = r\sum_{\boldsymbol{k}} p_{\boldsymbol{k}} \frac{(k_1k_2\bar{k}_1\bar{k}_2)^r}{((k_1\bar{k}_2)^r + (\bar{k}_1k_2)^r)^2}.
$$

Thus, $\mathrm{E}[X^*] = 2\mathrm{E}[X_1^*]$. Next, we show that for any $k_1, k_2$,

$$
\frac{(k_1k_2\bar{k}_1\bar{k}_2)^r}{((k_1\bar{k}_2)^r + (\bar{k}_1k_2)^r)^2} \le \frac{1}{4}.
$$

Using the fact that $((k_1\bar{k}_2)^r - (\bar{k}_1k_2)^r)^2 \ge 0$,

$$
\begin{aligned}
0 &\le (k_1\bar{k}_2)^{2r} + (\bar{k}_1k_2)^{2r} - 2(k_1k_2\bar{k}_1\bar{k}_2)^r \\
&= ((k_1\bar{k}_2)^r + (\bar{k}_1k_2)^r)^2 - 4(k_1k_2\bar{k}_1\bar{k}_2)^r.
\end{aligned}
$$

Rearranging yields the desired result. Furthermore, the inequality is strict if $k_1\bar{k}_2 \ne \bar{k}_1k_2$. Then, taking expectations over all possible $\boldsymbol{k} = (k_1, k_2)$,

$$
\mathrm{E}[X^*] = 2r\sum_{\boldsymbol{k}} p_{\boldsymbol{k}} \frac{(k_1k_2\bar{k}_1\bar{k}_2)^r}{((k_1\bar{k}_2)^r + (\bar{k}_1k_2)^r)^2} \le 2r\sum_{\boldsymbol{k}} p_{\boldsymbol{k}}\frac{1}{4} = \frac{r}{2},
$$

with strict inequality if there is some $\boldsymbol{k}$ such that $k_1\bar{k}_2 \ne \bar{k}_1k_2$ and $p_{\boldsymbol{k}} > 0$. Thus, equality is reached if $p_{\boldsymbol{k}} > 0$ only for points $(k_1, k_2)$ with $\frac{k_1}{k_2} = \frac{\bar{k}_1}{\bar{k}_2}$; that is, it must be that $K_1 = aK_2$ for some $a > 0$. ■