Contests between groups of unknown size

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Abstract

We consider group contests where the number of competing groups is fixed but group sizes are stochastic and unobservable to contest participants at the time of investment. We allow for arbitrary correlation between group sizes. When the distribution of group sizes is symmetric, the symmetric equilibrium aggregate investment is always lower than in a symmetric group contest where the same expected group size is commonly known. The same holds for asymmetric distributions of group sizes in contests between two groups. The reduction in investment due to population uncertainty is stronger the larger the variance in appropriately defined relative group impacts. When group sizes are independent conditional on a common shock, a stochastic increase in the common shock mitigates the effect of group size uncertainty unless the common and idiosyncratic components of group size are strong complements.

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1 Introduction

Contests are a major mechanism of non-market allocation of resources. Examples include rent-seeking activities, such as companies fighting for government contracts, or lobbyists promoting legislation, as well as litigation, political campaigns and R&D competition (for a review see, e.g., Congleton, Hillman and Konrad, 2008). In many cases, contests involve *groups* of independent actors competing for a common goal in order to secure a non-rival prize for all members of the winning group. For example, loosely defined groups of US telecommunication giants (such as Comcast and Verizon) and Internet content providers (the likes of Netflix, Amazon, Microsoft, Google and Facebook) find themselves on opposite sides of the net neutrality debate and lobby actively for their respective interests. Importantly, the number of players in each of the competing groups may not be exactly known, especially as many companies refrain from taking a public stance on the issue, while working covertly behind the scenes. The same applies to political campaigns where, especially after the US Supreme Court's Citizens United vs. Federal Election Commission decision, the number and identity of donors is easy to conceal. Yet, a standard assumption in most models of contests in the literature is that the number of competitors is commonly known. More recently, researchers have started to explore the effects of population uncertainty on behavior in contests between individuals (Münster, 2006; Myerson and Wärneryd, 2006; Lim and Matros, 2009; Fu, Jiao and Lu, 2011; Kahana and Klunover, 2015, 2016; Ryvkin and Drugov, 2017).[1] However, to the best of our knowledge, there is no study to date that examines the effects of population uncertainty on behavior in group contests.[2]

In this paper, we study contests between groups whose number is fixed but size is uncertain. This type of population uncertainty is different from the one arising in games (e.g., contests) between individuals where the number of competing units (players) is uncertain. In contrast, our setting is more akin to Bayesian games where players receive private signals about their own type but only know the distribution of others' types. Intuitively, by participating in a group contest, a player updates her beliefs about the size (and hence an appropriately defined measure of strength) of her own group. We allow

[1]Similarly, population uncertainty has been studied theoretically in auctions (McAfee and McMillan, 1987; Harstad, Kagel and Levin, 1990; Levin and Ozdenoren, 2004) and other environments, such as voting, coordination games and public goods (e.g., Myerson, 1998, 2000; Makris, 2008, 2009; De Sinopoli and Pimienta, 2009; Mohlin, Östling and Wang, 2015).

[2]There is a well-developed theoretical literature on group contests examining a wide range of environments with common knowledge about group sizes (e.g., Katz, Nitzan and Rosenberg, 1990; Nitzan, 1991; Baik, 1993; Riaz, Shogren and Johnson, 1995; Nti, 1998; Esteban and Ray, 2001; Baik, 2008; Nitzan and Ueda, 2009, 2011; Ryvkin, 2011; Lee, 2012; Chowdhury, Lee and Sheremeta, 2013; Kolmar and Rommeswinkel, 2013; Brookins and Ryvkin, 2016; Barbieri and Malueg, 2016).

for arbitrary correlations between group sizes; in particular, our model can accommodate common shocks to group sizes. For example, the results of the 2016 Presidential election in the United States and ensuing changes in the regulatory climate have created a common shock to the sizes of groups fighting on different sides of many contentious issues such as net neutrality, health care, and environmental regulation. Thus, a better understanding of how contest behavior responds to population uncertainty and to various structural or policy-induced shocks that can generate correlation (positive and negative) between group sizes, is important for a wide range of prominent social and political issues. In particular, we show that our results have implications for optimal disclosure policies when the contest designer has an option to commit to disclosure of the number of competitors in each group prior to investment decisions.

In our model, we consider group contest environments where the prize is non-rival, such that all members of the winning group enjoy the full benefits associated with the prize. We also assume that individual efforts are perfect substitutes within groups, such that competition between groups is accompanied by within-group incentives for free-riding. While there are some group contest environments in which the prize is (partly or fully) rival, or where individual efforts within groups are aggregated according to a different technology (e.g., weak-link, or best-shot), our assumptions closely describe the main examples mentioned above.

There are three main findings. First, for symmetric distributions of group sizes, we show that the symmetric equilibrium investment for any non-degenerate distribution with mean group size $\overline{k}$ is strictly lower than the symmetric equilibrium investment in a symmetric group contest where group size is fixed and commonly known to be $\overline{k}$. Furthermore, we show that the reduction in equilibrium investment increases with the variance of an appropriately defined measure of *relative group impact* for any given distribution. This is consistent with other results in the literature that link larger reductions in equilibrium investment to the degree of heterogeneity in contestants' ability or impact in the contest (Baik, 2008; Ryvkin, 2011).

Second, we show that when group sizes are symmetrically distributed and correlated so that the size of each group is an increasing function of a common component and an idiosyncratic component, an increase in the common component (in the usual stochastic order) leads to a reduction in the variance of relative group impact unless the common and idiosyncratic components are strong complements. Thus, in most cases a large positive shock to all group sizes mitigates the effect of population uncertainty on individual and aggregate investment in contests.

Third, we show that in the case of two competing groups with (possibly) asymmetrically distributed group sizes with means $\bar{k}_1$ and $\bar{k}_2$, aggregate equilibrium investment is lower than in the corresponding contest where the group sizes are fixed and commonly known to be $\bar{k}_1$ and $\bar{k}_2$.

Relation to previous literature

There are several previous studies that explore *individual contests* with a stochastic number of players. Myerson and Wärneryd (2006) show that aggregate equilibrium investment in an uncertain contest with mean number of players equal to μ is strictly lower than in a contest where the number of players is equal to μ with certainty. This finding is very similar to our first main result concerning the negative effects of group size uncertainty on (individual and aggregate) equilibrium investment in the group contest setting. Münster (2006) considers a similar environment to Myerson and Wärneryd (2006) and examines the robustness of the reduction in rent dissipation to various alternative factors, such as the assumption that contestants are risk averse, or that they face budget caps. In a related setting, Lim and Matros (2009) consider contests in which the number of players is a random variable drawn from the binomial distribution with parameters (n, q). They show, first, that equilibrium investment is nonmonotone and single-peaked in the probability of participation q whenever the number of potential players $n > 2$, and second, that investment is nonmonotone in n, provided q is not too large. Furthermore, in the same spirit as our first result and consistent with Myerson and Wärneryd (2006), they show that aggregate equilibrium investment is lower than in a corresponding contest with certain group size, nq. Another research thread that relates to the study of population uncertainty explores the effect of disclosure of the number of participating players on aggregate effort. Fu, Jiao and Lu (2011) show that the optimality of disclosure depends on the properties of the impact function for the generalized lottery-form contest success function. While the aforementioned studies concentrate on Tullock contests and their lottery-form generalizations, Ryvkin and Drugov (2017) derive a set of general results for a broader class of winner-take-all rank-order tournaments, including new results for tournaments with a stochastic number of players that encompass the existing results for Tullock contests.

Our paper adds a new environment to the literature on (general) games with population uncertainty that also includes auctions (McAfee and McMillan, 1987; Harstad, Kagel and Levin, 1990; Levin and Ozdenoren, 2004), elections and voting games (Myerson, 1998, 2000), and binary public good provision (Makris, 2008). Furthermore, our contribution is novel in that the manifestation of population uncertainty in group contests is different from that in individual contests. Specifically, while the previous literature concerns uncertainty about the number of competing units, the uncertainty in our setting relates to the size of the (fixed) number of competing units. This represents an additional dimension through which the (general) notion of population uncertainty can affect equilibrium behavior in competitive environments.

Finally, our results also relate to the theoretical literature on contests (both between individuals and between groups) with incomplete information about players' types. In our setting, uncertainty about group sizes can be interpreted as a form of uncertainty about the groups' relative "strengths." Earlier work on contests with private information includes Hurley and Shogren (1998) and Malueg and Yates (2004). Fey (2008) established existence and provided a (numerical) characterization of the equilibrium bidding function for a two-player contest with uniformly distributed marginal effort costs. Ryvkin (2010) extended the analysis to contests with more than two players, general cost distributions, and a more general class of contest success functions. More recent work has further demonstrated the existence and, in some cases, uniqueness, of pure strategy Bayesian equilibrium for individual contests (see, e.g., Wasser, 2013*a*,*b*; Ewerhart, 2014; Einy et al., 2015; Ewerhart and Quartieri, 2016), and for group contests with incomplete information (Brookins and Ryvkin, 2016; Barbieri and Malueg, 2016). Existing results suggest that private information can sometimes reduce and sometimes increase aggregate equilibrium investment compared to the full information case (Wasser, 2013*a*). In contrast, in our setting, we show that incomplete information concerning group sizes unambiguously leads to a reduction in aggregate equilibrium investment.

The rest of the paper is organized as follows. In Section 2, we set up the model and characterize the equilibrium. Section 3 provides general results and examples for symmetric distributions of group sizes. In Section 4, we consider contests between two groups and allow for arbitrary (possibly asymmetric) distributions of group sizes. Section 5 contains a discussion and concluding remarks.

2 The model

2.1 Preliminaries

Consider a contest between $n \geq 2$ groups indexed by $i = 1, \ldots, n$. The number of players in each group i, denoted by K_i, is a random variable drawn from set $M_i = \{1, 2, \ldots, m_i\}$, where $m_i \geq 1$ can be finite or infinite.[3] We will use $\boldsymbol{K} = (K_1, \ldots, K_n)$ to denote the

[3]We assume that the minimal number of players in each group is one; that is, we do not consider a scenario in which some groups may not exist at all. Such a setting would conflate the effects of uncertainty with respect to group sizes with those of uncertainty about the number of competing units. The latter

random vector of group sizes with support $M = \times_{i=1}^{n} M_i$, and $\boldsymbol{k} \in M$ to denote a generic realization of $\boldsymbol{K}$. Let $\mathbf{1} \in M$ denote a vector of ones, and $\geq$ denote the usual component-wise partial order (when applied to vectors). We use $p_{\boldsymbol{k}} = \Pr(\boldsymbol{K} = \boldsymbol{k})$ to denote the joint probability mass function (pmf) of group sizes. We allow for the possibility that group sizes are not independent; that is, $p_{\boldsymbol{k}}$ is not necessarily equal to the product of marginal pmfs p_{k_i}.

All participating players simultaneously choose investment levels $x_{ij} \in \mathbb{R}_+$, where x_{ij} denotes the investment of player j in group i (referred to as "player ij"). The total investment of group i, denoted by X_i, is the sum of individual investments: $X_i = \sum_{j=1}^{K_i} x_{ij}$.

We consider a group contest with the contest success function (CSF) of the lottery form (Tullock, 1980) where each group's impact function is homogeneous of degree $r \in (0, 1]$. Thus, the probability that group i wins the contest conditional on $\boldsymbol{K}$ is given by

$$P_i(X_i, X_{-i} | \boldsymbol{K} = \boldsymbol{k}) = \begin{cases} \frac{1}{n}, & \text{if } X_1 = \ldots = X_n = 0 \\ \frac{X_i^r}{\sum\limits_{l=1}^{n} X_l^r}, & \text{otherwise.} \end{cases} \tag{1}$$

All players in the winning group receive a prize normalized to one, while players in other groups receive zero prize. All players are risk-neutral expected payoff maximizers.

2.2 Equilibrium investment

In our setting, participating players do not observe the realization of $\boldsymbol{K}$ at the time of investment. From an outsider's perspective, $\sum_{\boldsymbol{k} \in M} p_{\boldsymbol{k}} \phi(\boldsymbol{k})$ then gives the expectation of some function $\phi(\boldsymbol{k})$ with respect to the joint distribution of group sizes. From the perspective of a participating player, however, the distribution of the vector of group sizes is updated (cf., e.g., Harstad, Kagel and Levin, 1990; Myerson and Wärneryd, 2006).

Let I_{ij} denote a random variable equal to 1 if player j is selected to participate in the contest as a member of group i, and equal to zero otherwise. Using Bayes' rule, player ij should update the probability of any vector of group sizes $\boldsymbol{k}$ to

$$\tilde{p}_{\boldsymbol{k}}^i \equiv \Pr(\boldsymbol{K} = \boldsymbol{k} | I_{ij} = 1) = \frac{\Pr(I_{ij} = 1 | \boldsymbol{K} = \boldsymbol{k}) p_{\boldsymbol{k}}}{\sum_{\boldsymbol{l} \in M} \Pr(I_{ij} = 1 | \boldsymbol{K} = \boldsymbol{l}) p_{\boldsymbol{l}}}.$$

effects have been explored extensively in the literature on individual contests with size uncertainty (see, e.g., Münster, 2006; Myerson and Wärneryd, 2006; Lim and Matros, 2009; Ryvkin and Drugov, 2017). In this paper, we focus on the effects of group size uncertainty in group contests and keep the number of groups fixed.

Given that players are equally likely to be selected as participants, it follows that

$$\tilde{p}^i_{\boldsymbol{k}} = \frac{k_i p_{\boldsymbol{k}}}{\bar{k}_i}, \tag{2}$$

where $\bar{k}_i = \sum_{\boldsymbol{l} \in M} l_i p_{\boldsymbol{l}}$ is the (prior) expected number of players in group i.[4]

It then follows from (2) that from the perspective of a participating player in group i expectations are updated as $\frac{1}{\bar{k}_i} \sum_{\boldsymbol{k} \in M} p_{\boldsymbol{k}} k_i \phi(\boldsymbol{k})$. The payoff of player ij, conditional on being selected, is, therefore,

$$\pi_{ij} = \frac{1}{\bar{k}_i} \sum_{\boldsymbol{k} \in M} p_{\boldsymbol{k}} \frac{k_i X_i^r}{\sum_{h=1}^n X_h^r} - x_{ij}.$$

We will study the properties of a semi-symmetric equilibrium in pure strategies, where all players in group i choose the same investment x_i^*. Assuming all participating players other than ij choose such investment levels, the payoff of player ij from some deviation investment x_{ij} is

$$\pi_{ij}(x_{ij}, (x_h^*)_{h=1}^n) = \frac{1}{\bar{k}_i} \sum_{\boldsymbol{k} \in M} p_{\boldsymbol{k}} \frac{k_i (x_{ij} + (k_i - 1) x_i^*)^r}{(x_{ij} + (k_i - 1) x_i^*)^r + \sum_{h \neq i} (k_h x_h^*)^r} - x_{ij}. \tag{3}$$

The first-order conditions $\frac{\partial \pi_{ij}}{\partial x_{ij}} = 0$ evaluated at $x_{ij} = x_i^*$ produce the system of equations

$$\frac{r}{\bar{k}_i} \sum_{\boldsymbol{k} \in M} p_{\boldsymbol{k}} \frac{k_i^r (x_i^*)^{r-1} \sum_{h \neq i} (k_h x_h^*)^r}{(\sum_{h=1}^n (k_h x_h^*)^r)^2} = 1, \quad i = 1, \ldots, n. \tag{4}$$

The following proposition shows that any interior solution to this system of equations is a Nash equilibrium in the contest. All proofs are relegated to Appendix A.

Proposition 1 *For $r \in (0, 1]$, if $(x_i^*)_{i=1}^n$ is an interior solution to the system of equations (4), then it is a semi-symmetric Nash equilibrium.*

Note that, in general, there need not be an interior solution (or even a solution at all) to such a nonlinear system of equations, and there may be no pure strategy equilibrium in the contest. Nevertheless, while we cannot guarantee equilibrium existence for all possible

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[4] Consider the probability that $I_{ij} = 1$ given $\boldsymbol{K} = \boldsymbol{k}$. First, player j is equally likely to be selected as one of the total number of participants, $\sum_{h=1}^n k_h$. Thus, the probability that j is a participant at all is $\frac{\sum_{h=1}^n k_h}{\sum_{h=1}^n m_h}$. Second, the probability that player j is a member of group i, given that she is a participant at all is equal to the number of players in group i, divided by the total number of participants, $\frac{k_i}{\sum_{h=1}^n k_h}$. Together, these observations imply that $\Pr(I_{ij} = 1 | \boldsymbol{K} = \boldsymbol{k}) = \frac{k_i}{\sum_{h=1}^n m_h}$.

group size distributions, Proposition 1 implies existence (by construction) for situations when the existence of an interior solution to (4) can be established.

Below we derive and study the solution (and hence, the equilibrium) for two important and relevant cases. First, we derive the fully symmetric equilibrium investment level for arbitrary n under the assumption that the distribution of group sizes is symmetric. Second, we derive the semi-symmetric equilibrium investment levels for arbitrary (possibly asymmetric) distributions in the case where there are $n = 2$ groups.

3 Symmetric group size distributions

Consider the case where the distribution of group sizes is symmetric, i.e., $p_{\boldsymbol{k}} = p_{\rho(\boldsymbol{k})}$ for any permutation ρ of the components of $\boldsymbol{k}$. In this case, we look for a fully symmetric equilibrium, with $x_i^* = x^*$ for all $i = 1, \ldots, n$. Let $\bar{k} = \bar{k}_i$ denote the symmetric expected group size, and $m = m_i$ the symmetric maximum group size. The system of Eqns. (4) simplifies to

$$x^* = \frac{r}{\bar{k}} \sum_{\boldsymbol{k}} p_{\boldsymbol{k}} \frac{k_i^r \sum_{h \neq i} k_h^r}{(\sum_{h=1}^n k_h^r)^2}. \tag{5}$$

Let $S_i = \frac{K_i^r}{\sum_{h=1}^n K_h^r}$ denote the (random) *relative impact* of group i. Due to the symmetry of the joint distribution, since $\sum_{i=1}^n S_i = 1$, the expected relative impact of each group is $\mathrm{E}[S_i] = \frac{1}{n}$. Moreover, from the definition of variance, $\mathrm{E}[S_i^2] = \mathrm{Var}[S_i] + \frac{1}{n^2}$. Equation (5) then can be written in the form

$$x^* = \frac{r}{\bar{k}} \left(\frac{n-1}{n^2} - \mathrm{Var}[S_i] \right). \tag{6}$$

Equation (6) has a very intuitive structure. First, in the degenerate case when $\mathrm{Var}[S_i] = 0$ it collapses into the well-known expression for the symmetric equilibrium investment in a group contest where all group sizes are fixed and equal to $\bar{k}$,

$$x^0 = \frac{r(n-1)}{\bar{k} n^2}. \tag{7}$$

Second, when group sizes are stochastic, $\mathrm{Var}[S_i] \geq 0$. Moreover, the variance is strictly positive unless group sizes are perfectly correlated. Therefore, we obtain the following result.

Proposition 2 *For symmetrically distributed group sizes, the symmetric equilibrium investment is not higher than in the case when the group size is fixed at $\bar{k}$; that is, $x^* \leq x^0$. Moreover, the inequality is strict unless (i) $p_{\boldsymbol{k}}$ is degenerate or (ii) $p_{\boldsymbol{k}} = 0$ for all $\boldsymbol{k} \neq \alpha \mathbf{1}$,* *for some* $\alpha \in \{1, \ldots, m\}$.

It also follows from Eq. (6) that the reduction in equilibrium investment due to uncertainty is stronger the larger the variance in relative group impacts.

We will now explore the role of dependence between group sizes. One way to model such dependence is to assume that the size of group i is given by $K_i = g(Z, Y_i)$, where Z is an integer random variable common for all groups, Y_i are i.i.d. integer idiosyncratic shocks independent of Z, and g is an integer-valued function increasing in both arguments. One simple example is an additive model, $g(z, y_i) = z+y_i$, where the common and idiosyncratic components are perfect substitutes. Intuitively, as the size of the common component Z increases, variation in relative group impacts should go down because, for any given realization of Z, group sizes become more similar. This intuition may not work, however, when there is strong complementarity between Z and Y_i, because an increase in Z can lead to an increase in the effect of Y_i on K_i and hence to a larger variation in relative impacts. The result is summarized in the following proposition.

Proposition 3 *Suppose* $\frac{g(z+1,y)}{g(z,y)}$ *is decreasing (increasing) in* y*. Then a stochastic increase in* Z *leads to a lower (higher)* $\mathrm{Var}[S_i]$.

Returning to the example discussed above, for $g(z, y_i) = z+y_i$ we have $\frac{g(z+1,y)}{g(z,y)} = \frac{z+1+y}{z+y}$ decreasing in y and hence Proposition 3 indeed implies that a stochastic increase in Z will lead to a reduction in $\mathrm{Var}[S_i]$. In fact, this example is a special case of a more general property.

Corollary 1 *Suppose* $g(z, y_i) = \phi(a(z) + b(y_i))$ *where* $a(\cdot)$ *and* $b(\cdot)$ *are increasing and* $\phi(\cdot)$ *is increasing and log-concave (log-convex). Then a stochastic increase in* Z *leads to a lower (higher)* $\mathrm{Var}[S_i]$.

Corollary 1 provides a straightforward way to construct examples where a stochastic increase in Z leads to an increase in $\mathrm{Var}[S_i]$. For example, function $\phi(t) = 2^{t^2}$ is log-convex, and hence $g(z, y_i) = 2^{(z+y_i)^2}$ produces the desired result. Another example (not covered by Corollary 1, but easily verified via Proposition 3) is $g(z, y_i) = z^{y_i}$. As expected, in both cases, the common and idiosyncratic components are strong complements.

Next, we consider expected total investment in the contest, $\mathrm{E}[X^*] = n\bar{k}x^*$. Equation (6) gives

$$\mathrm{E}[X^*] = r\left(\frac{n-1}{n} - n\mathrm{Var}[S_i]\right). \tag{8}$$

Note that, for individual investment x^*, a stochastic increase in Z has two effects. On the one hand, according to Proposition 3, when $\frac{g(z+1,y)}{g(z,y)}$ is decreasing in y, it leads to a lower $\mathrm{Var}[S_i]$, thereby increasing x^*. On the other hand, a stochastic increase in Z also increases $\bar{k}$, thereby reducing x^*. In contrast, expected total investment in the contest is independent of $\bar{k}$. Thus, a stochastic increase in Z only affects expected total investment in the contest through its effect on the variance of relative group impacts.

Corollary 2 *(i) Expected total investment in the contest is decreasing in* $\mathrm{Var}[S_i]$.
(ii) Suppose $K_i = g(Z, Y_i)$ *as in Proposition 3 and* $\frac{g(z+1,y)}{g(z,y)}$ *is decreasing (increasing) in* y*. Then a stochastic increase in* Z *leads to a higher (lower) expected total investment in the contest.*

In contests with population uncertainty, the contest designer may be able to disclose the number of participants; it is, therefore, of interest to explore whether commitment to such disclosure is optimal. Parallel results have been established in the literature on contests between individuals. Lim and Matros (2009) showed that disclosure leads to an increase in *ex ante* expected aggregate investment in Tullock contests with the binomial distribution of the number of players. Fu, Jiao and Lu (2011) extended this result to contests with a CSF of the generalized lottery form and showed that disclosure can increase or decrease aggregate investment depending on the shape of the CSF's impact function. Ryvkin and Drugov (2017) further generalized these results to arbitrary tournaments with arbitrary distributions of the number of players.

In our setting, disclosing the number of players in each group will generate the same total equilibrium group investment $X_i^0 = k_i x_i^0 = \frac{r(n-1)}{n^2}$ in all groups, where $x_i^0 = \frac{r(n-1)}{k_i n^2}$ is the semi-symmetric equilibrium effort level in the corresponding group contest with commonly known group sizes $\boldsymbol{k}$ (Baik, 1993). The resulting aggregate contest investment, $X^0 = \frac{r(n-1)}{n}$, exceeds the expected total investment without disclosure, Eq. (8), in all but degenerate cases.

Corollary 3 *The disclosure of group sizes leads to an increase in expected total investment. The effect is strictly positive with the exception of the degenerate cases in Proposition 2.*

Next, we provide an example to illustrate the results presented in this section, using a symmetric multivariate distribution with dependence between group sizes.

3.1 Example 1: Symmetric distribution

Consider the additive model described above, with $K_i = Z + Y_i$. Further, assume that Z is a Poisson random variable with parameter θ, and Y_i, $i = 1, \ldots, n$, are i.i.d. zero-truncated Poisson random variables with parameter λ. The pmf for the zero-truncated

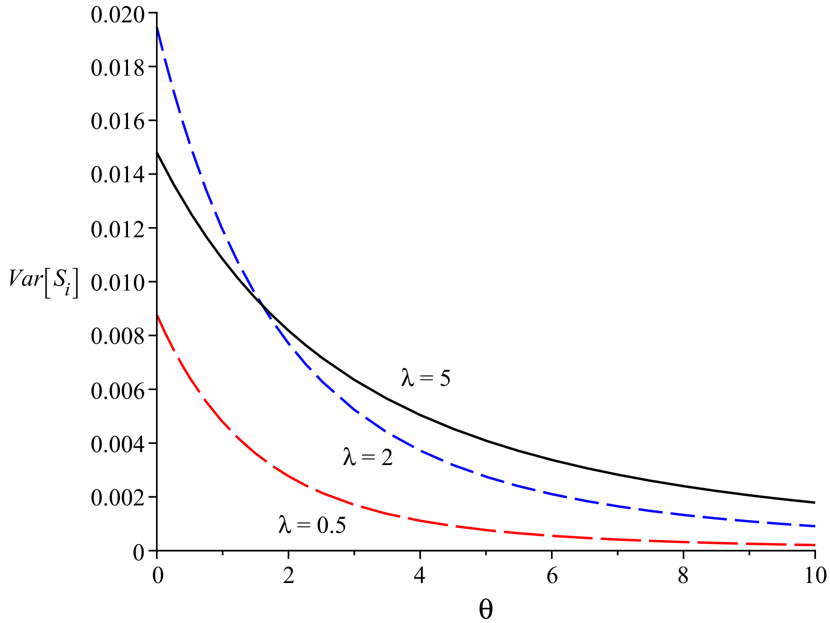

Figure 1: Variance of S_i for $\lambda \in \{0.5, 2, 5\}$ and θ varying from 0 to 10.

Poisson distribution is given by

$$\Pr[Y_i = y | Y_i \geq 1] = \frac{\lambda^y}{y!(e^\lambda - 1)}. \tag{9}$$

The joint pmf of $\boldsymbol{K}$ is, therefore,[5]

$$p_{\boldsymbol{k}} = \frac{e^{-\theta} \lambda^{\sum_{i=1}^n k_i}}{(e^\lambda - 1)^n \prod_{i=1}^n k_i!} \sum_{s=0}^{\min\{k_1,\ldots,k_n\}-1} (s!)^{n-1} \left(\frac{\theta}{\lambda^n}\right)^s \prod_{i=1}^n \binom{k_i}{s}, \quad \boldsymbol{k} \geq \mathbf{1}. \tag{10}$$

Since $g(Z, Y_i)$ is additive, $\frac{g(z+1,y)}{g(z,y)}$ is decreasing in y. Thus, by Proposition 3, a stochastic increase in Z will lead to a reduction in $\mathrm{Var}[S_i]$. For the Poisson distribution, an increase in θ generates a stochastic increase in Z.

To illustrate the main results from this section, we consider the case of $n = 3$ groups. For simplicity, we set $r = 1$. We compute $\mathrm{Var}[S_i]$ directly using (10) and the definition of S_i. Figure 1 plots $\mathrm{Var}[S_i]$ as a function of θ for three different values of λ. For each case, $\lambda \in \{0.5, 2, 5\}$, the variance of relative group impact is strictly decreasing as θ increases. Furthermore, as $\theta \to \infty$, $\mathrm{Var}[S_i] \to 0$.

The effects of θ and $\mathrm{Var}[S_i]$ on equilibrium individual investment are also highlighted

[5]This joint pmf can be derived using the same approach as to deriving the standard multivariate Poisson distribution in which the Y_i, $i = 1, \ldots, n$ are not truncated at zero, (cf., e.g., Johnson, Kotz and Balakrishnan, 1997).

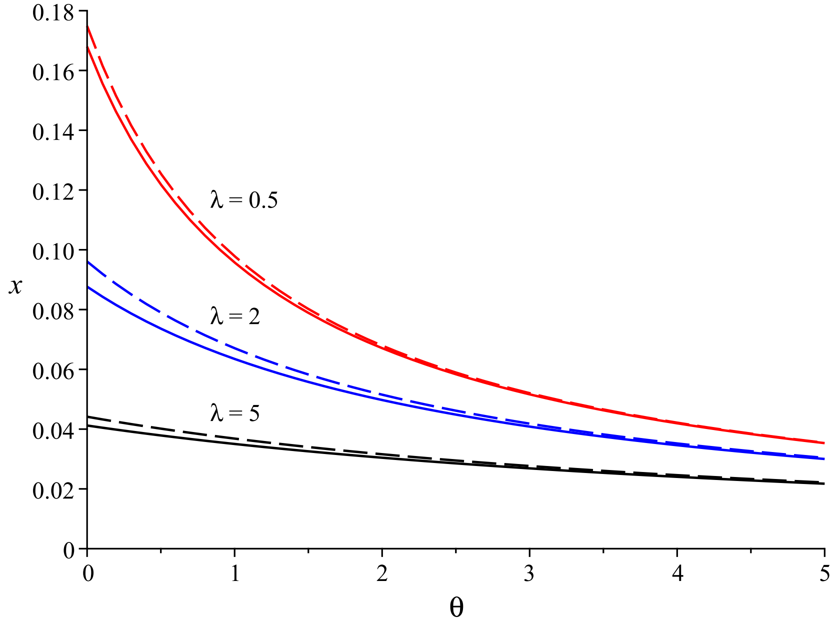

Figure 2: Equilibrium individual investment, $x^*(\theta, \lambda)$ (solid lines), compared with $x^0(\theta, \lambda)$ (dashed lines), the corresponding equilibrium investment in a group contest where all group sizes are fixed and equal to $\bar{k}(\theta, \lambda)$.

in Figure 2, which plots the equilibrium investment under group size uncertainty alongside the corresponding equilibrium investment in a deterministic contest with all group sizes fixed and equal to $\bar{k}$. For the multivariate distribution (10) with $n = 3$, the mean group size as a function of θ and λ is given by

$$\bar{k}(\theta, \lambda) = \theta + \frac{\lambda e^{\lambda}}{e^{\lambda} - 1}.$$

To make clear the relevant comparison, we denote by $x^0(\theta, \lambda)$ the equilibrium investment in the corresponding deterministic contest with $\bar{k}(\theta, \lambda)$ active participants in each group. As expected, $x^*(\theta, \lambda) < x^0(\theta, \lambda)$, for each (θ, λ). However, keeping λ fixed, as θ increases, the difference between the equilibrium investment with and without group size uncertainty disappears.

4 Arbitrary group size distributions with $n = 2$

In this section, we consider the case where there are only $n = 2$ groups, but the joint distribution over group sizes need not be symmetric. The first-order conditions (4) take the form

$$\sum_{\boldsymbol{k}} p_{\boldsymbol{k}} \frac{k_1^r (x_1^*)^{r-1} (k_2 x_2^*)^r}{((k_1 x_1^*)^r + (k_2 x_2^*)^r)^2} = \frac{\bar{k}_1}{r}, \quad \sum_{\boldsymbol{k}} p_{\boldsymbol{k}} \frac{k_2^r (x_2^*)^{r-1} (k_1 x_1^*)^r}{((k_1 x_1^*)^r + (k_2 x_2^*)^r)^2} = \frac{\bar{k}_2}{r}, \tag{11}$$

which immediately implies

$$\frac{x_2^*}{x_1^*} = \frac{\bar{k}_1}{\bar{k}_2}. \tag{12}$$

That is, the ratio of equilibrium investment levels for the individual members of group 2 relative to group 1 is equal to the inverse ratio of the expected number of players in group 2 relative to group 1. Furthermore, this implies that the *expected* equilibrium group level investment, $\mathrm{E}[X_i^*] = \bar{k}_i x_i^*$, is identical across groups.

Using (12) and (11), we obtain

$$x_i^* = \frac{r}{\bar{k}_i} \sum_{\boldsymbol{k}} p_{\boldsymbol{k}} \frac{(k_1 k_2 \bar{k}_1 \bar{k}_2)^r}{(k_1^r \bar{k}_2^r + \bar{k}_1^r k_2^r)^2}. \tag{13}$$

First, notice that for a group contest with deterministic group sizes equal to $\bar{k}_1$ and $\bar{k}_2$, equation (13) reduces to

$$x_i^0 = \frac{r}{4\bar{k}_i}, \tag{14}$$

which corresponds to the equilibrium derived in Baik (1993, 2008). Second, it follows that in equilibrium, total expected investment in the contest, $\mathrm{E}[X^*] = \mathrm{E}[X_1^* + X_2^*] = 2\mathrm{E}[X_1^*]$, is given by

$$\mathrm{E}[X^*] = 2r \sum_{\boldsymbol{k}} p_{\boldsymbol{k}} \frac{(k_1 k_2 \bar{k}_1 \bar{k}_2)^r}{(k_1^r \bar{k}_2^r + \bar{k}_1^r k_2^r)^2}. \tag{15}$$

As seen from (15), $\mathrm{E}[X^*] = \frac{r}{2}$ if $K_1 = aK_2$ for some $a > 0$ or if both K_1 and K_2 are degenerate (even if they are different). However, in all other non-deterministic cases, expected total investment is lower.

Proposition 4 *For $n = 2$, with an arbitrary distribution of group sizes, expected total investment is lower as compared to the case when the group sizes are fixed at $(\bar{k}_1, \bar{k}_2)$; that is, $\mathrm{E}[X^*] \leq \frac{r}{2}$. The inequality is strict unless $p_{\boldsymbol{k}}$ is degenerate or $K_1 = aK_2$ for some $a > 0$.*

Similar to the case of symmetric group size distributions (cf. Corollary 3), Proposition 4 informs on the consequences of disclosure of group sizes (k_1, k_2). Total equilibrium investment with disclosure, $X^0 = \frac{r}{2}$, exceeds expected total investment without disclosure in all but the degenerate cases.

Corollary 4 *For $n = 2$, with an arbitrary distribution of group sizes, the disclosure of group sizes leads to an increase in expected total investment. The effect is strictly positive with the exception of the degenerate cases in Proposition 4.*

We illustrate these results using two examples. Example 2 uses an asymmetric distribution constructed using Poisson random variables (following an approach similar to the one used for Example 1). This example illustrates the effect of positive correlation between group sizes. Example 3 considers the case in which there is always a fixed total number of active players m, divided between the two groups according to a Binomial distribution, in order to illustrate the effect of negative correlation between group sizes.

4.1 Example 2: Asymmetric distribution with positive correlation

Similar to the construction in Example 1, let $\boldsymbol{K} = (K_1, K_2)$ be given by $K_i = Z + Y_i$, where Z is a Poisson random variable with parameter θ, and Y_i, $i = 1, 2$, are independent zero-truncated Poisson random variables with (possibly different) parameters λ_i. The joint pmf of $\boldsymbol{K}$ is given by

$$p_{\boldsymbol{k}} = \frac{e^{-\theta}\lambda_1^{k_1}\lambda_2^{k_2}}{(e^{\lambda_1}-1)(e^{\lambda_2}-1)k_1!k_2!} \sum_{s=0}^{\min\{k_1,k_2\}-1} \binom{k_1}{s}\binom{k_2}{s} s! \left(\frac{\theta}{\lambda_1\lambda_2}\right)^s, \quad \boldsymbol{k} \geq \boldsymbol{1}. \tag{16}$$

Note that the mean number of active players in group $i = 1, 2$ as a function of θ and λ_i is

$$\bar{k}_i(\theta, \lambda_i) = \theta + \frac{\lambda_i e^{\lambda_i}}{e^{\lambda_i} - 1}. \tag{17}$$

Again, for simplicity, we set $r = 1$. Then, using equation (13), we compute $\mathrm{E}[X^*]$ for different combinations of parameters $(\theta, \lambda_1, \lambda_2)$.

For any fixed pair of parameters (λ_1, λ_2), as θ increases, Z becomes more important and the idiosyncratic components, Y_i, $i = 1, 2$, become less important for the realized group size. Consequently, realizations of $\boldsymbol{K}$ with $K_1 = K_2$ will become relatively more likely. For these realizations of $\boldsymbol{K}$, the term in the summand of equation (13) is equal to $\frac{p_{\boldsymbol{k}}}{4}$. Thus, intuition suggests that as θ increases, these terms receive greater probability weight, and $\mathrm{E}[X^*]$ will tend to increase. Although this argument seems intuitive, it does not always hold, as we show in the examples below.

In Figure 3, we plot $\mathrm{E}[X^*]$ for $(\lambda_1, \lambda_2) = (2, 5)$ and for $(\lambda_1, \lambda_2) = (8, 5)$, with θ varying from 0 to 10. In both cases, expected total investment is below the corresponding total investment in a contest with deterministic group sizes, which is represented by the

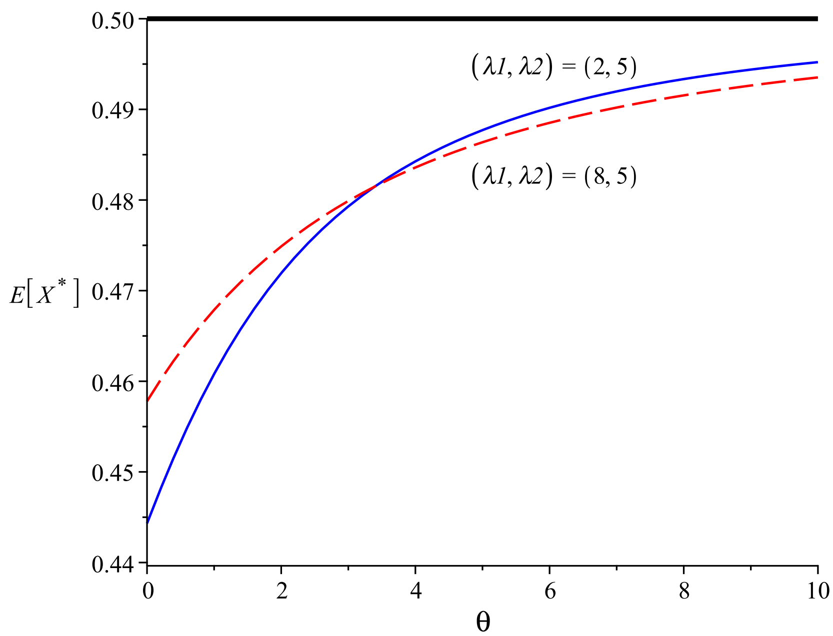

Figure 3: Expected total investment $\mathrm{E}[X^*]$ for $(\lambda_1, \lambda_2) = (2, 5)$ and $(\lambda_1, \lambda_2) = (8, 5)$.

horizontal reference line at 0.5. Furthermore, as θ increases, expected total investment is monotonically increasing, consistent with the preceding intuition.

In contrast, Figure 4 provides an example where, if the asymmetry between groups is strong enough and one of the groups has a sufficiently low parameter λ_i, expected total investment may not be monotonically increasing in θ. Using $(\lambda_1, \lambda_2) = (1, 15)$ and $(\lambda_1, \lambda_2) = (0.5, 15)$, Figure 4 shows that $\mathrm{E}[X^*]$ is, at first, decreasing in θ, then subsequently increasing in θ. The source of this nonmonotonicity in $\mathrm{E}[X^*]$ with respect to θ is the fact that we use zero-truncated Poisson random variables for the idiosyncratic components, Y_i. Specifically, if λ_2 is relatively large, while λ_1 is sufficiently small, the truncated distribution for Y_1 is substantially different from its standard Poisson distribution, while the truncated distribution for Y_2 is very similar to its standard Poisson distribution. This differential impact of truncation on the distributions of Y_1 and Y_2 then distorts the relative likelihood of realizations in which group sizes are the same, provided θ is also sufficiently small. Nevertheless, even in these somewhat unusual cases, once θ grows sufficiently large, $\mathrm{E}[X^*]$ is increasing in θ, as can be observed in Figure 4.

4.2 Example 3: Negative correlation between group sizes

In this example, we consider the effects of negative correlation between group sizes on the equilibrium investment in the contest. Suppose there are m potential participants

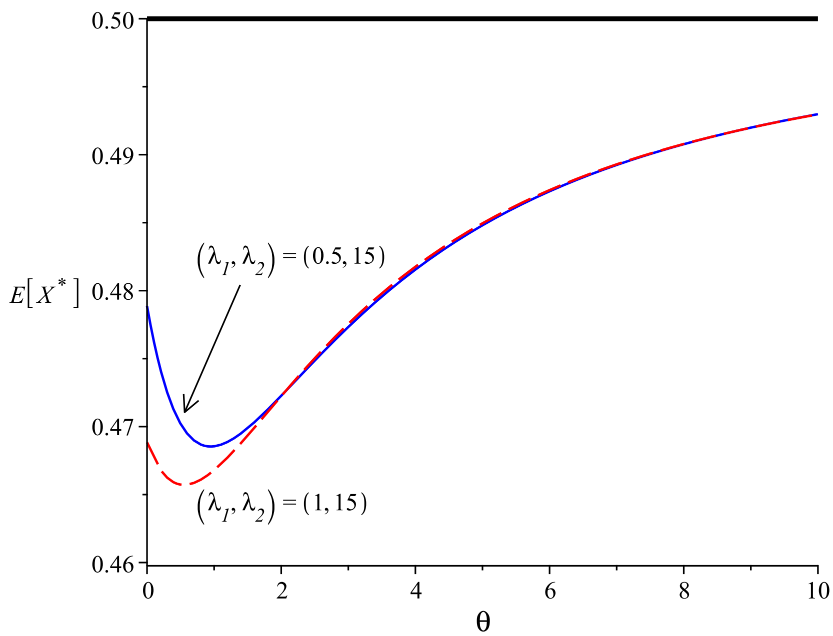

Figure 4: Expected total investment $\mathrm{E}[X^*]$ for $(\lambda_1, \lambda_2) = (0.5, 15)$ and $(\lambda_1, \lambda_2) = (1, 15)$.

in the population. Each potential participant is active in one of the two groups. Let $q \in [0, 1]$ be the probability of any given player being a member of group 1. In this setting, $K_1 + K_2 = m$, from which the perfect negative correlation between group sizes is evident. Then the probability of $\boldsymbol{K} = \boldsymbol{k} = (k_1, m-k_1)$ is given by the binomial probability

$$p^B_{\boldsymbol{k}} = \binom{m}{k_1} q^{k_1}(1-q)^{m-k_1}, \quad k_1 = 0, \ldots, m. \tag{18}$$

Once again, since we assume the minimum group size in any group is 1, we use a truncated distribution, updated to ensure that $k_1 \geq 1$ and $k_1 \leq m-1$ (corresponding to $k_2 \geq 1$). The resulting pmf is given by

$$p_{\boldsymbol{k}} = \frac{\binom{m}{k_1} q^{k_1}(1-q)^{m-k_1}}{1-(q^m+(1-q)^m)}, \quad k_1 = 1, \ldots, m-1. \tag{19}$$

The resulting mean group sizes are given by

$$\bar{k}_1 = mq\left(\frac{1-q^{m-1}}{1-(q^m+(1-q)^m)}\right), \quad \bar{k}_2 = m(1-q)\left(\frac{1-(1-q)^{m-1}}{1-(q^m+(1-q)^m)}\right).$$

Fixing $r = 1$ and using equation (13), we compute $\mathrm{E}[X^*]$ for various parameters (m, q). In Figure 5 we plot $\mathrm{E}[X^*]$ against q for $m \in \{5, 10, 20, 50, 500\}$. Several features are worth highlighting. First, as predicted, expected total investment is below the corresponding

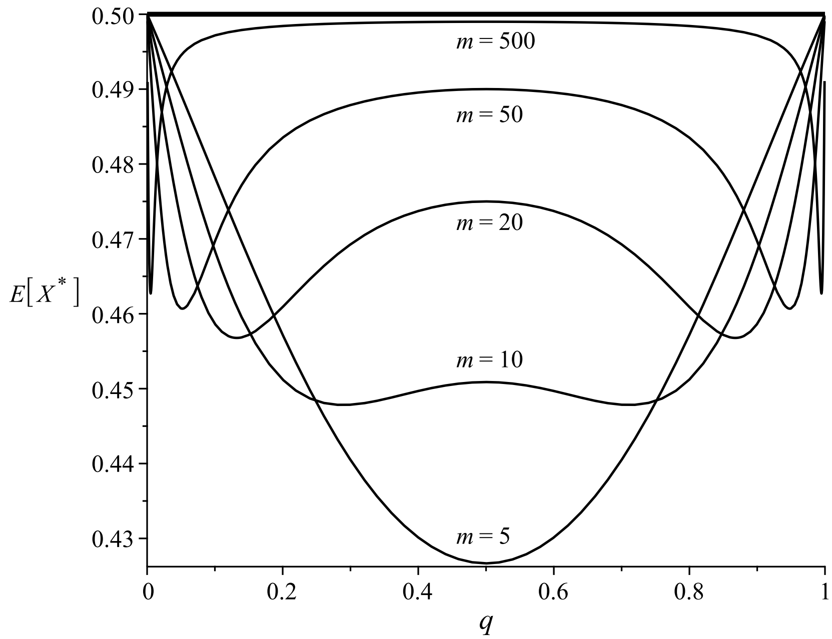

Figure 5: Expected total investment $\mathrm{E}[X^*]$ as a function of q, for $m \in \{5, 10, 20, 50, 500\}$. The reference line at 0.50 indicates equilibrium total investment in a contest without group size uncertainty.

total investment in a contest without group size uncertainty. Second, when q is equal to 0.5 (so that each group is equally likely to receive a particular participant), expected total investment is increasing in the population size. For q close to 0.5, this ordering is preserved. Intuitively, when q is close to 0.5, even if there is perfect negative correlation, realized group sizes are much more likely to be the same, especially when the number of potential participants is very large.

However, when there are stronger asymmetries between groups, i.e., when q is closer to 0 or closer to 1, expected total investment may be higher for smaller populations than for larger ones. For example, as shown in Figure 5, when $q = 0.1$ (or $q = 0.9$), $\mathrm{E}[X^*]$ is higher for $m = 5$ than for $m = 10$ or $m = 20$.

5 Discussion and conclusion

In this paper, we provide the first investigation of group contests under population uncertainty. More specifically, we study group contests where the number of groups is fixed, but the sizes of the groups are unknown. We consider the simplest, canonical group contest setting in which the prize awarded to the winning group is non-rival and the efforts of individuals are perfect substitutes within groups. Individuals do not observe the size of their own group or of any other groups at the time of investment. Rather, group sizes are determined stochastically, according to a general distribution. In particular, we allow for arbitrary correlations between group sizes, including those driven by a common shock.

We first characterize the semi-symmetric equilibrium in pure strategies for the general model, then provide three main results under additional assumptions. Our first main result illustrates that for symmetric distributions of group sizes, the symmetric equilibrium investment for any non-degenerate distribution with mean group sizes $\bar{k}$ is strictly lower than in a group contest where group sizes are fixed and commonly known to be $\bar{k}$. That is, population uncertainty (in terms of the sizes of the groups) lowers the individual (and aggregate) equilibrium investment. This finding is similar to findings regarding the effects of population uncertainty in *individual* contests derived by Myerson and Wärneryd (2006) and Lim and Matros (2009). In addition, we show that the reduction in equilibrium investment is driven by the variance of *relative group impact*, which is a property derived from the (symmetric) distribution of group sizes. One can think of this notion of relative group impact as a (random) measure of the group's strength in the contest, in which case our result implies that the reduction in equilibrium investment is stronger when the distribution generates higher variance in the groups' strengths. This finding echoes some of the results on the effects of heterogeneity on aggregate effort in contests where it is generally believed that larger asymmetries lead to lower effort.[6]

Our second main result, extending on the first, concerns the effect of correlation between group sizes when the distribution is symmetric. To this end, we consider the case in which the size of each group is an increasing function of a common component and an idiosyncratic component. We show that, provided the common and idiosyncratic components are not sufficiently strong complements, a stochastic increase in the common component reduces the variance of relative group impact, thereby mitigating the effects of population uncertainty on equilibrium investment.

We then consider the case of two competing groups with a distribution of group sizes that need not be symmetric. Our third result shows that, as for symmetric distributions, aggregate equilibrium investment is lower under population uncertainty than in the corresponding contest with group sizes fixed and commonly known to be the respective means of the (possibly) asymmetric distribution.

Altogether, our results suggest that population uncertainty has a negative effect on equilibrium investment in group contests. One immediate implication is the effect of disclosure of group sizes. These findings are compatible with those derived for individual

[6]See, however, Ryvkin (2013) and Drugov and Ryvkin (2017) who show that this "common wisdom" is far from universal.

contests with population uncertainty, despite the fact that in our setting, population uncertainty manifests in a manner more akin to uncertainty about the relative strength of the groups (due to variations in the group's size) than about the number of competing units. In addition, we show that, depending on the nature and the extent of the correlations between group sizes, this effect may be magnified or moderated when there is a common shock, such as a change in regulatory policy, the approach of an election, or a landmark judicial decision.

Our investigation also motivates some promising avenues for future work. For example, while our analysis assumes that the distribution of group sizes is stochastic, the true source of such group size uncertainty may be the endogenous entry decisions of potential participants with commonly aligned interests. Any investigation of endogenous entry into group contests, or the formation of groups and alliances in contests, has the potential to generate population uncertainty for participants at the time of investment. Our results provide a general and substantially simpler framework in which to consider the effects of such population uncertainty (however determined) on behavior. However, it seems equally appealing to explore the potential sources of uncertainty in a model that specifically incorporates endogenous entry by potential participants.

Another interesting extension would be to consider a group contest in which the prize is partially or fully rival, or in which the individual efforts of group members are aggregated according to a different production technology. For instance, previous studies have examined the weak-link and best-shot mechanisms in group contests under the standard assumption that group sizes are fixed and commonly known. Another natural extension is to consider heterogeneous participants. Finally, while our results inform on the consequence of (non)disclosure of the sizes of competing groups by the contest designer, our study raises additional questions about whether or when it may be optimal for participants to conceal or reveal their own participation in a group contest. For instance, are organizations that conceal their lobbying efforts from the public eye acting optimally, or could they improve their expected payoff by publicly declaring their support (or intended support) to encourage increased participation? Our investigation provides a framework for the analysis of these kinds of questions.

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A Proofs

Proof of Proposition 1 It is sufficient to show that the payoff function $\pi_{ij}(x_{ij}, (x_h^*)_{h=1}^n)$ in Eq. (3) is globally strictly concave in x_{ij}. This follows immediately from the observation that function $\frac{x}{x+a}$ is strictly concave in x for any $a > 0$, function $(x+b)^r$ is concave in x for any $b \geq 0$ and $r \in (0, 1]$, and a composition of a strictly concave and concave functions is strictly concave. It then follows that if $(x_i^*)_{i=1}^n$ is an interior solution to the system of n equations in (4), setting $x_{ij} = x_i^*$ is a best response for player ij, and hence $(x_i^*)_{i=1}^n$ is a Nash equilibrium. ■

Proof of Proposition 3 Since $\text{Var}[S_i] = \text{E}[S_i^2] + \frac{1}{n^2}$, we will show that $\text{E}[S_i^2]$ behaves as stated in the proposition. By the law of iterated expectations, $\text{E}[S_i^2] = \text{E}_Z[\text{E}_{\boldsymbol{Y}}[S_i^2|Z]]$, where the inner expectation is taken over the realizations of $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ for a given Z and the outer expectation is with respect to Z. In order to prove the result stated in the proposition, it is sufficient to establish that $\text{E}_{\boldsymbol{Y}}[S_i^2|Z = z]$ is decreasing (increasing) in z when $\frac{g(z+1,y)}{g(z,y)}$ is decreasing (increasing). Using the definition of S_i and symmetry, we have

$$\text{E}_{\boldsymbol{Y}}[S_i^2|Z = z+1] - \text{E}_{\boldsymbol{Y}}[S_i^2|Z = z] = \frac{1}{n}\sum_i \left(\text{E}_{\boldsymbol{Y}}[S_i^2|Z = z+1] - \text{E}_{\boldsymbol{Y}}[S_i^2|Z = z]\right)$$

$$= \frac{1}{n}\text{E}_{\boldsymbol{Y}} \sum_i \left(\frac{g(z+1, Y_i)^2}{(\sum_j g(z+1, Y_j))^2} - \frac{g(z, Y_i)^2}{(\sum_j g(z, Y_j))^2} \right) = \frac{1}{n}\text{E}_{\boldsymbol{Y}} A_1(z, \boldsymbol{Y}) + \frac{1}{n}\text{E}_{\boldsymbol{Y}} A_2(z, \boldsymbol{Y}),$$

where

$$A_1(z, \boldsymbol{y}) = \sum_i \frac{g(z+1, y_i)}{\sum_j g(z+1, y_j)} \left(\frac{g(z+1, y_i)}{\sum_j g(z+1, y_j)} - \frac{g(z, y_i)}{\sum_j g(z, y_j)} \right),$$

$$A_2(z, \boldsymbol{y}) = \sum_i \frac{g(z, y_i)}{\sum_j g(z, y_j)} \left(\frac{g(z+1, y_i)}{\sum_j g(z+1, y_j)} - \frac{g(z, y_i)}{\sum_j g(z, y_j)} \right).$$

We will establish that $A_1(z, \boldsymbol{y}) \leq (\geq) 0$ and $A_2(z, \boldsymbol{y}) \leq (\geq) 0$ provided $\frac{g(z+1,y)}{g(z,y)}$ is decreasing (increasing) in y. The sign of $A_1(z, \boldsymbol{y})$ is determined by the sign of the expression

$$B_1(z, \boldsymbol{y}) = \sum_i g(z+1, y_i) \left[g(z+1, y_i) \sum_j g(z, y_j) - g(z, y_i) \sum_j g(z+1, y_j) \right]$$

$$= \sum_{i,j} g(z+1, y_i) g(z, y_i) g(z, y_j) \left[\frac{g(z+1, y_i)}{g(z, y_i)} - \frac{g(z+1, y_j)}{g(z, y_j)} \right]$$

Suppose, for concreteness, that $\frac{g(z+1,y)}{g(z,y)}$ is decreasing in y. Then

$$\sum_{i,j}[g(z+1,y_i)-g(z+1,y_j)]g(z,y_i)g(z,y_j)\left[\frac{g(z+1,y_i)}{g(z,y_i)}-\frac{g(z+1,y_j)}{g(z,y_j)}\right]\leq 0,$$

because the two expressions in square brackets have opposite signs. This gives

$$\begin{aligned}B_1(z,\boldsymbol{y}) &= \sum_{i,j} g(z+1,y_i)g(z,y_i)g(z,y_j)\left[\frac{g(z+1,y_i)}{g(z,y_i)}-\frac{g(z+1,y_j)}{g(z,y_j)}\right]\\ &\leq \sum_{i,j} g(z+1,y_j)g(z,y_i)g(z,y_j)\left[\frac{g(z+1,y_i)}{g(z,y_i)}-\frac{g(z+1,y_j)}{g(z,y_j)}\right]\\ &= \sum_{j,i} g(z+1,y_i)g(z,y_j)g(z,y_i)\left[\frac{g(z+1,y_j)}{g(z,y_j)}-\frac{g(z+1,y_i)}{g(z,y_i)}\right] = -B_1(z,\boldsymbol{y}),\end{aligned}$$

which implies $B_1(z,\boldsymbol{y})\leq 0$. The derivation for $A_2(z,\boldsymbol{y})$ and the case when $\frac{g(z+1,y)}{g(z,y)}$ is increasing in y is similar. ■

Proof of Corollary 1 Suppose, for concreteness, that ϕ is log-concave. This implies that $\frac{\phi(t+x)}{\phi(x)}$ is decreasing in x for any $t\geq 0$. Letting $x=x_2+y$ and $t=x_1-x_2+y$, where $x_1\geq x_2$, it follows that $\frac{\phi(x_1+y)}{\phi(x_2+y)}$ is decreasing in y for any $x_1\geq x_2$. Therefore, for any $y_1\geq y_2$ we have $\frac{\phi(x_1+y_1)}{\phi(x_2+y_1)}\leq\frac{\phi(x_1+y_2)}{\phi(x_2+y_2)}$. Setting $x_1=a(z+1)$, $x_2=a(z)$, $y_1=b(y+1)$ and $y_2=b(y)$, obtain that $\frac{g(z+1,y)}{g(z,y)}$ is decreasing in y. The case of log-convex ϕ is similar. ■

Proof of Proposition 4 From equation (13),

$$\mathrm{E}[X_1^*]=\mathrm{E}[X_2^*]=r\sum_{\boldsymbol{k}}p_{\boldsymbol{k}}\frac{(k_1k_2\bar{k}_1\bar{k}_2)^r}{((k_1\bar{k}_2)^r+(\bar{k}_1k_2)^r)^2}.$$

Thus, $\mathrm{E}[X^*]=2\mathrm{E}[X_1^*]$. Next, we show that for any k_1,k_2,

$$\frac{(k_1k_2\bar{k}_1\bar{k}_2)^r}{((k_1\bar{k}_2)^r+(\bar{k}_1k_2)^r)^2}\leq\frac{1}{4}.$$

Using the fact that $((k_1\bar{k}_2)^r-(\bar{k}_1k_2)^r)^2\geq 0$,

$$\begin{aligned}0 &\leq (k_1\bar{k}_2)^{2r}+(\bar{k}_1k_2)^{2r}-2(k_1k_2\bar{k}_1\bar{k}_2)^r\\ &= ((k_1\bar{k}_2)^r+(\bar{k}_1k_2)^r)^2-4(k_1k_2\bar{k}_1\bar{k}_2)^r.\end{aligned}$$

Rearranging yields the desired result. Furthermore, the inequality is strict if $k_1\bar{k}_2\neq\bar{k}_1k_2$.

Then, taking expectations over all possible $\boldsymbol{k} = (k_1, k_2)$,

$$\mathrm{E}[X^*] = 2r \sum_{\boldsymbol{k}} p_{\boldsymbol{k}} \frac{(k_1 k_2 \bar{k}_1 \bar{k}_2)^r}{((k_1 \bar{k}_2)^r + (\bar{k}_1 k_2)^r)^2} \leq 2r \sum_{\boldsymbol{k}} p_{\boldsymbol{k}} \frac{1}{4} = \frac{r}{2},$$

with strict inequality if there is some $\boldsymbol{k}$ such that $k_1 \bar{k}_2 \neq \bar{k}_1 k_2$ and $p_{\boldsymbol{k}} > 0$. Thus, equality is reached if $p_{\boldsymbol{k}} > 0$ only for points (k_1, k_2) with $\frac{k_1}{k_2} = \frac{\bar{k}_1}{\bar{k}_2}$; that is, it must be that $K_1 = aK_2$ for some $a > 0$. ■