Competition in a Posted-Salary Matching Market under Private Information*

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Abstract

We study a posted-salary labor market in which firms engage in salary competition. Firms' preferences over workers are private information, creating uncertainty about competitive pressure for different workers. We consider a baseline 2-firm, 2-worker model, then extend the analysis to larger markets by replicating the baseline. We characterize the unique Bayesian-Nash equilibrium, in which each firm type chooses a distributional strategy with interval support in the salary space. The main result shows that competition is localized, in the sense that firm types with a common most preferred worker choose non-overlapping, adjacent supports. We also provide numerical results to show that the equilibrium strategies in finite replicated markets converge to the corresponding equilibrium strategies in a market with a continuum of firms and workers.

Keywords: Salary competition; posted salary; two-sided markets; bayesian games; replicated markets

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1 Introduction

In many labor markets, firms compete with each other for workers along several dimensions. These include salary, employee benefits, bonuses, health insurance coverage, and opportunities for career advancement. In most cases, firms and workers bargain over these terms of employment during the hiring process. But in some situations, firms pre-commit to the salary or compensation package to be offered for a particular job opening. It is not unusual for a firm to set policies regarding benefits, bonuses, health plans, or vacation time, rather than personalize the terms of employment for each individual employee. The terms of employment may be inflexibile due to firm-wide policies, or contractual arrangements, or because the salary for the position has been widely advertized. Or perhaps a department within an organization must negotiate, in advance, with a board of directors or a department manager, over the precise compensation package to be offered for the position. Thus, there are interesting labor market environments in which firms cannot tailor the various terms of employment to the particular worker who they hire.

In this article, we analyze the competitive behavior of firms in one such environment. We consider a posted-salary labor market in which the firms possess private information about their own preferences. The market consists of a set of firms, each with one available position, and a set of workers, each searching for one position. Each firm in the market posts a fixed salary and commits to pay the posted salary to whichever worker is eventually hired. Moreover, the firms' preferences over the set of workers are private information, so that each firm faces uncertainty about the level of demand (among their competitors) for their more preferred candidates.

The combination of private information with the firms' commitments to posted salaries creates a novel, albeit specific environment. In particular, when the firms are unable to renegotiate with each candidate, the possibility of Pareto inefficient matchings emerges as a concern. For example, if a firm makes a high salary offer that is rejected by their first preference, any subsequent offer to a less preferred candidate must be made at the same high salary, which may preclude otherwise profitable arrangements. Alternatively, if a firm makes a lower salary offer to their second choice, then discovers that their first choice would have accepted a marginally higher salary offer, they cannot renege and recontract even when it would be profitable to do so.

The timing of events for the environment studied in this article are as follows. Before firms and workers are matched, the firms choose salaries privately and simultaneously. The matching process is modeled using the firm-proposing deferred acceptance algorithm, developed by Gale & Shapley (1962). Thus, the process begins with each firm extending an offer at their posted salary to at most one worker, and each worker then tentatively accepts at most one offer and rejects all others. Any firm whose offer is rejected can then extend the *same* salary offer to another worker, after which the workers again decide which (if any) offer to tentatively accept. The process continues until no new offers are made, at which time all tentatively accepted offers are confirmed. In addition, there are two important rules that apply to the matching process. First, firms cannot revisit workers who have previously rejected their offer. Second, firms cannot retract an offer that has been tentatively accepted.

A well-known result from the matching literature is that the firm-proposing deferred acceptance algorithm gives each firm a dominant strategy to make offers in a straightforward manner. For the matching procedure described above, this means that firms have a dominant strategy to make offers in order of preference, but only to workers who are acceptable to the firm at its chosen posted salary. Furthermore, since all workers care only about salary, there is a unique stable matching for any profile of firm preferences and salary offers. As a result, no worker has an incentive to strategically reject an offer. Thus, without any incentives for strategic sequencing of offers (by firms) or strategic rejection of offers (by workers), the focus in this article is on the strategic salary decisions made by the firms in the presence of private information.

The article first analyzes a two-firm, two-worker model, in which the firms can be one of four (preference) types. The first result proves that, in the two-firm, two-worker model, there are no pure strategy equilibria. However, there does exist a unique Bayesian Nash equilibrium in mixed (distributional) strategies which are continuous with interval support. The second and third results derive the characterization of this unique Bayesian Nash equilibrium. Proposition 2 establishes that the equilibrium exhibits a *separation of types*, in the sense that between two types who have a common most preferred worker, one type mixes over salary offers that are everywhere higher than the salaries chosen by the other type. The type that offers a lower equilibrium support 'concedes defeat' in the event that the other firm is the type offering higher salaries. Instead, those firms concentrate on competing in the event that their competitor is of their own type. Similarly, each firm of the type offering the higher salaries pays a premium just high enough to always outbid the other type, then concentrates on competing against their own type by mixing over an interval of salaries. In this respect, competition is localized to firms with 'similar' types. The full characterization result is provided in Proposition 3, where we show that the relative marginal value attached to the workers by different types determines which type makes the higher offers in equilibrium.

In the second part of the article, we extend the analysis to larger markets by replicating the two-firm, two-worker model. In the limit, when there are a continuum of firms and a continuum of workers, there is no aggregate uncertainty about the realization of types. Thus, competition in equilibrium is confined to the most popular worker class. The third main result is a characterization of the equilibrium in the limit case. We then prove the existence of a Bayesian Nash equilibrium in continuous distributional strategies with interval support for each finite replicated market. The proof, which is by construction, establishes that the *s*eparation result obtained in Proposition 2 also applies to types with a common most preferred worker class in finite replicated markets.

Finally, we provide numerical results to show that the finite market equilibrium strategies converge to the corresponding continuum equilibrium strategies as the number of replications approaches infinity. Thus, as the replicated markets get larger, the uncertainty over competitive pressure disappears, and competitive behavior is only sustained for the most popular worker class.

1.1 Related Literature

A similar environment with posted salaries has been studied by Burdett and Mortensen (1998). They consider a game where a continuum of firms choose permanent wage offers and a continuum of workers search by sequentially sampling from the set of offers. Workers search both while unemployed and while employed for a job with an acceptable, or higher wage, respectively. The principal result in Burdett and Mortensen (1998) is that wage dispersion is a robust outcome when workers must search for individual offers, provided that workers search while employed as well as when unemployed. They characterize the unique equilibrium (steady state) distribution of wage offers under different assumptions about firm and worker heterogeneity.

Our approach in this article differs in several important respects from the setup used in Burdett and Mortensen (1998). First, workers are not identical in our model. Offers are posted and then directed by firms to particular workers, rather than posted for workers to search for and accept as they please. In particular, this allows for a firm to exclude workers who are not acceptable to them at their posted salary. Second, we assume that firms have only unit demands. That is, each firm wants to be matched with just one worker, rather than to build up a team of workers.

The most important implication of these differences is that, in our model, each worker faces a potentially different distribution of offers. Firms have different ordinal preferences and may control which workers are allowed to accept their offer. In terms of our results, we also find that the equilibrium salary distributions offered by different firm types exhibit wage dispersion (both within and among the types). However, the wage dispersion is driven by private information among the firms and the competitive pressure among heterogeneous firms for heterogeneous workers. This is distinct from the wage dispersion derived in Burdett and Mortensen (1998), which is driven in part by the multi-unit demands of the firms, and in part by their heterogeneous productivities with respect to a set of perfectly substitutable workers.

This article is also related to several others that study salary competition in two-sided markets, although almost all of them investigate markets with *complete information*. In particular, Bulow and Levin (2006), Niederle (2007), and Kojima (2007) study the effects of a centralized matching mechnism on salaries relative to the competitive equilibrium.[1] Bulow and Levin (2006) provide a theoretical result which suggests that worker salaries are compressed and depressed relative to the competitive equilibrium, by the centralized matching algorithm used in the National Resident Matching Program (NRMP).

The first treatment of matching with salaries is Shapley and Shubik (1972), which modifies Gale and Shapley (1962) to incorporate a transferrable utility good in which salaries can be paid. The early literature on matching with transferable utility was further developed by Crawford and Knoer (1981) and subsequently, by Kelso and Crawford (1982), who devised a salary adjustment process which converges to a core allocation.[2] More recently, Hatfield and Milgrom (2005) develop a model of matching with contracts that incorporates the Kelso and Crawford (1982) model.[3] They show that if the preferences of the firms satisfy a *gross substitutes* condition and a *law of aggregate demand* condition, then truthful reporting is a dominant strategy for workers in a worker-proposing matching mechanism.

While matching with salaries has attracted considerable interest in recent years, the existing literature omits a study of the environment in which firms' preference orderings over workers are private information. The study that comes closest to doing so is by Hoppe et al. (2009), who introduce a model of assortative matching in which there is incomplete information on both sides of the market. However, the incomplete information in their setting relates to attributes of potential partners, rather than preference orderings of potential competitors. Both workers and firms can send signals regarding their own attributes. However, once a worker has chosen a signal in their model, every firm has the same ranking over workers, based on their signals. Likewise, once a firm has chosen a signal, every worker has the same ranking over firms.

In contrast, the objective of this study is to understand the effects of private information about the preference orderings of potential competitors on salary competition. The baseline model is most similar to the Bulow and Levin (2006) setup. However, it extends their approach by allowing firms to have different primitive preferences over workers. This assumption then allows for firms' preferences to be private information, which is the main innovation to the existing literature on matching with salaries.

The rest of the article is organized as follows. Section 2 describes the model and the matching process, then introduces the general two-firm, two-worker game with private information. In Section 3, we prove that there are no pure strategy equilibria, then provide an example which

[1] Another paper that investigates the importance of various assumptions in the Bulow and Levin (2006) model is Gonzalez-Diaz and Siegel (Forthcoming), who focus on a set of job features, including salaries, reputation, responsibility, work hours, training, and quality of facilities, that may affect a hospital's attractiveness to workers in a non-linear manner.

[2] A core allocation in this context is a one-to-one matching along with a salary schedule, in which no firm and no worker can negotiate a salary at which they would prefer each other over their current partners at their current salaries.

[3] Other related work includes Hoppe, Moldovanu, and Sela (2009), Crawford (2008),

demonstrates the main features of the equilibrium in distributional strategies. Section 3 also establishes the *separation* result, and provides the full characterization for the symmetric Bayesian Nash equilibrium in continuous distributional strategies with interval support. In Section 4, the model is extended to larger markets. We characterize the equilibria of the limit case in which there are a continuum of firms and workers, then prove existence (and the separation result) for finite replicated markets. Finally, Section 5 presents the numerical results, which show that the equilibrium strategies in finite replicated markets converge to the corresponding continuum equilibrium strategies as the number of replications approaches infinity. Section 6 concludes.

2 A Two-Firm, Two-Worker Model

Suppose there are two firms $f_1, f_2 \in F$ and two workers $w_1, w_2 \in W$. Each firm has strict preferences over the set $\{w_1, w_2, \emptyset\}$, where $\emptyset$ represents being unmatched. It is safe to ignore any preference ranking in which remaining single is the most preferred option, since a firm with those preferences will exit the market. Thus, there are four possible preference rankings for each firm.

$$\begin{array}{llllcllll} P_a: & w_1 & w_2 & \emptyset & \qquad & P_b: & w_1 & \emptyset & w_2 \\ P_c: & w_2 & w_1 & \emptyset & & P_d: & w_2 & \emptyset & w_1. \end{array}$$

Assume that each preference ranking is represented by a pair of values, one for each worker, while the value of remaining unmatched is 0. This assumption is somewhat restrictive, since it means that two firms with same preference ranking also have the same values for the workers. In Section 6, I discuss ways to relax this assumption about the type space.

Refer to a firm with preferences P_k as a firm of type k. Then the set of firm types is described as $\mathscr{P}^f = \{a, b, c, d\}$ where, for example, $a = (a_1, a_2)$ and a_j is the value of worker j to type a for each $j = 1, 2$. In order to represent the preference rankings, the values of the different types must satisfy the following conditions.

$$\begin{array}{lcl} a_1 > a_2 > 0 & \qquad & b_1 > 0 > b_2 \\ c_2 > c_1 > 0 & & d_2 > 0 > d_1. \end{array}$$

Definition 1. *A worker w is* **acceptable** *to firm f if f prefers w to remaining unmatched.*

The second definition modifies the standard notion of an acceptable worker to account for the preferences of the firms at a given salary level.

Definition 2. *Given any salary, x_f, chosen by firm f, a worker w is* **salary-acceptable** *to firm f if f's value for worker w is greater than x_f.*

The values corresponding to each type are common knowledge, however, each firm knows only its own type. The types are drawn independently according to a common prior distribution π over $\mathscr{P}^f = \{a, b, c, d\}$. Given the two disjoint sets of agents, define a matching as follows.

Definition 3. *A matching is a function $\mu : F \cup W \to F \cup W \cup \{\emptyset\}$ such that*

(1) $\mu(f) \in W \cup \{\emptyset\}$ *for all* $f \in F$,
(2) $\mu(w) \in F \cup \{\emptyset\}$ *for all* $w \in W$, *and*
(3) $\mu(\mu(i)) = i$ *for all* $i \in F \cup W$ *with* $\mu(i) \neq \{\emptyset\}$.

Let $\mathscr{M}$ denote the set of all matchings.

For any firm f with type $k = (k_1, k_2)$, the utility derived from a matching $\mu \in \mathscr{M}$ is given by

$$u_k^f(\mu) \quad = \quad \begin{cases} k_1 & \text{if } \mu(f) = w_1 \\ k_2 & \text{if } \mu(f) = w_2 \\ 0 & \text{if } \mu(f) = \{\emptyset\} \end{cases} \quad .$$

Before the matching is determined, the firms each choose a salary. Then the following steps determine the matching outcome.

Step 1. Each firm makes an offer to (at most) one worker;
Step 2. Each worker tentatively accepts at most one offer, and rejects all others;
Step 3. Any firm whose offer was rejected may make the same salary offer to the other worker (who has not already rejected them);
Step 4. Each worker tentatively accepts at most one offer out of the one (if any) it tentatively holds, and the new offers received at *Step* 3, and rejects all others.
Step 5. Since there can be no new offers after *Step* 4, the procedure terminates and all tentative matches are confirmed.

In principle, both the firms and workers could adopt many different strategies. However, as the following two remarks make clear, it is not necessary to consider anything more than the simplest strategies.

Remark 1. *For any set of chosen salaries, each firm has a dominant strategy to make offers in order of preference to salary-acceptable workers only.*

Once firms have chosen salaries, the matching procedure described by *Steps 1 to 5* is equivalent to the Gale-Shapley Deferred Acceptance (DA) algorithm for a specific matching market in which the firms' "preferences" are their original preferences, restricted to their (respective) sets of salary-acceptable workers, and workers' preferences are given by ranking the firms according to salary, from highest to lowest. It follows from Theorem 5 in Roth (1982), that firms have a dominant strategy to make offers in order of preferences, but restricted to salary-acceptable workers.

Remark 2. *For any profile of firm preferences and any set of chosen salaries, each worker has a dominant strategy to reject all but the highest salary offered to them.*

Recall that the firm-proposing DA mechanism is stable. That is, for any profile of reported preferences, it produces a matching that is stable with respect to the reported preferences. Since all the workers have the same preferences, there is a unique stable matching for each realization of firm preferences and set of chosen salaries. By Theorem 4.16 in Roth and Sotomayor (1990), every set of worker strategies that form a Nash equilibrium with the truthful strategies of the firms produces a matching that is stable with respect to the true preferences. Then since each induced market has a unique stable matching, and the matching mechanism is stable, there is no other Nash equilibrium strategy that dominates truth-telling by the workers, for any realization of firm types and posted salaries.

Since there are no incentives for strategic sequencing of offers by the firms, or for strategic rejection by the workers, the rest of the paper focuses on the behavior of the firms when they decide upon a salary. In fact, it is useful to describe the outcomes from the matching process by a direct revelation outcome function g. Let $g : \mathscr{P} \times \mathbb{R}_+^2 \to \mathscr{M}$ be an outcome function that maps the preferences (types) of the two firms and the salaries chosen by the firms into the set of matchings. Firms have a dominant strategy to announce their true preferences over salary-acceptable workers and the workers simply reject all but the highest offer made to them.

3 Equilibria in the Two-Firm, Two-Worker Model

The game is formally defined by $\Gamma = \left(F, W, \mathscr{P}, \mathbb{R}_+, \pi, \mathbf{g}, \{u_k^f\}_{f,k}\right)$, which consists of the sets of firms $F = \{f_1, f_2\}$, and workers $W = \{w_1, w_2\}$, the firm type space $\mathscr{P}$, the space of possible salaries $\mathbb{R}_+$, and the type distribution π. The outcome function $\mathbf{g}$ represents the matching process described by *Steps 1 - 5*, and $\{u_k^f\}_{f,k}$ are the utility functions for each firm and each firm type over the set of matchings.

3.1 Pure Strategy Equilibria

A pure strategy for a firm f is a function $\mathbf{s}_f : \mathscr{P}^f \to \mathbb{R}_+$ which selects a salary for each possible firm type. Given a strategy $\mathbf{s}_{-f}$ for the other firm, firm f's expected payoff from announcing a salary x_f when its type is k is given by

$$\mathbb{E}U_k^f\left(x_f, \mathbf{s}_{-f},\right) = \sum_{p \in \mathscr{P}^{-f}} \pi(p) \cdot u_k^f\left[g\left(k, p, x_f, s_{-f}(p)\right)\right].$$

Proposition 1. *As long as π has full support, there is no pure strategy Bayesian Nash equilibrium to the game, Γ.*

Proof. See Appendix. □

The intuition for the proof is that two firms of the same type, who are willing to accept their second favorite worker, will 'bid' each other up to the marginal value for the preferred worker. But then each has an incentive to give up on the preferred worker and maximize their payoff by hiring the second worker at a salary of 0. Once one firm does this, the other again has an incentive to offer slightly more than 0 for the preferred worker and the bidding war will begin again.

3.2 Distributional (Mixed) Strategy Equilibria

Formally, a distributional (or mixed) strategy for firm i is a function $\sigma_i : \mathscr{P} \to \Delta(\mathbb{R}_+)$ which announces, for each preference type, a distribution over salaries in $\mathbb{R}_+$. For simplicity, refer to the symmetric equilibrium (σ^*, σ^*) by the equilibrium strategy $\sigma^* = (G_a^*, G_b^*, G_c^*, G_d^*)$ where G_k^* is the cumulative distribution announced by a firm whose type is k. I assume that strategies are continuous distributions with interval support.[4] Before turning to the results, it is useful to work through a simple example for the two-firm, two-worker model.

Example 1. *Suppose $a = (2,1)$, $b = (2,-1)$, $c = (1,2)$, and $d = (-1,2)$, while $\pi(a) = \frac{1}{2}$, $\pi(b) = \frac{1}{8}$, $\pi(c) = \frac{1}{4}$, and $\pi(d) = \frac{1}{8}$.*

Notice that the marginal benefit to getting worker w_1 is higher for type b than type a, and the marginal benefit to getting worker w_2 is higher for type d than type c. Given these parameters, a natural conjecture is that type b firms will make higher offers than type a firms, and type d firms will make higher offers than type c firms. Furthermore, given the distribution of types π, worker

[4] There may be other types of symmetric equilibria, with non-interval support, or discontinuous strategies. In addition, there may be asymmetric equilibria.

w_1 is ex ante more popular (or believed to be more popular) than w_2. As such, one might expect to see higher salaries on average being offered to w_1.

Indeed, for Example 1, there exists an equilibrium, which is described as follows:

$$\begin{aligned} G_a^*(x) &= 2x \quad \text{on the support } \left[0, \frac{1}{2}\right] \\ G_b^*(x) &= \frac{7x - 3.5}{2 - x} \quad \text{on the support } \left[\frac{1}{2}, \frac{11}{16}\right] \\ G_c^*(x) &= 4x \quad \text{on the support } \left[0, \frac{1}{4}\right] \\ G_d^*(x) &= \frac{7x - 1.75}{2 - x} \quad \text{on the support } \left[\frac{1}{4}, \frac{15}{32}\right]. \end{aligned}$$

This equilibrium exhibits several interesting features. First, there is no overlap between the equilibrium supports of types with a common most preferred worker. Since the marginal value of getting worker w_1 is less for type a than for type b, firms of type b **always** announce higher salaries than firms of type a. In other words, firms of type a are resigned to getting their second favorite worker (w_2) when the other firm is type b. Instead, a type a firm focuses just on competing against another type a firm. On the other hand, a type b firm offers enough to ensure that it outbids any type a firm, then focuses on competing against the chance that the other firm is a type b. This type of‘separation result between types a and b is also exhibited by types c and d, and as will be shown below, is a characteristic of any equilibrium in continuous distributional strategies with interval support.

Second, equilibrium salaries are higher on average for firms of type a than type c and for type b than type d, even though they have comparable values for their respective preferences. This reflects the relative popularity of worker w_1 over worker w_2. This notion of popularity is manifested in the differences in the probabilities of facing another firm with the same most preferred worker. For types a and b, the probability of facing another type a or b is $\frac{5}{8}$, while for types c and d, the probability of facing another type c or d is only $\frac{3}{8}$. As a result, the average salaries offered in equilibrium are higher for type a than type c, and higher for type b than type d.

3.3 The Characterization

Consider the general two-firm, two-worker model. The salary strategies chosen by types that share a common most preferred worker together determine the matching for firms of those two types. On the other hand, salaries do not affect the matching output when the realized types do not have a common most preferred worker. Thus, pairs of types with common most preferred workers can be considered in isolation from one another. Without loss of generality, consider types a and b. The following two lemmas allow us to characterize the supports for the equilibrium strategies.

Lemma 1. *Between types with a common most preferred worker, the lowest salary offered in equilibrium must be 0.*

Proof. See Appendix. □

Lemma 2. *In equilibrium, there are no gaps between the equilibrium supports for types with a common most preferred worker.*

Proof. See Appendix. □

These two lemmas imply that equilibria must be consistent with one of four cases. In each case, type a mixes over $[\underline{x}_a, \overline{x}_a]$, and type b mixes over $[\underline{x}_b, \overline{x}_b]$, where

Case 1. $0 = \underline{x}_a < \underline{x}_b \leq \overline{x}_a < \overline{x}_b$
Case 2. $0 = \underline{x}_b < \underline{x}_a \leq \overline{x}_b < \overline{x}_a$
Case 3. $[\underline{x}_a, \overline{x}_a] \subset [\underline{x}_b, \overline{x}_b]$, and $\underline{x}_b = 0$
Case 4. $[\underline{x}_b, \overline{x}_b] \subset [\underline{x}_a, \overline{x}_a]$, and $\underline{x}_a = 0$.

Proposition 2 generalizes and formalizes the separation result illustrated in Example 1, by showing that there cannot be equilibria of the form described by *Case 3* or *Case 4*.

Proposition 2 (The Separation Result)**.** *Equilibrium supports do not overlap for types with a common most preferred worker. In particular then, any equilibrium must be of the form in Case 1 with* $\underline{x}_b = \overline{x}_a$ *or Case 2 with* $\underline{x}_a = \overline{x}_b$*.*

Proof. See Appendix. □

The proof for Proposition 2 is based on demonstrating that indifference cannot be satisfied simultaneously for both types on an interval with non-empty interior. As a result, the equilibrium supports in *Case 1* and *Case 2* must meet at their boundaries. For *Case 3* and *Case 4*, the same argument implies that a support which is a subset of the other must be a single point. Since best responses in pure strategies have already been ruled out, there must not exist an equilibrium in which one support is nested in the other.

The next proposition characterizes all symmetric Bayesian Nash equilibria in which strategies are continuous with interval support. Moreover, it provides the set of conditions that determine, for each pair of types with a common most preferred worker, whether their equilibrium supports are consistent with *Case 1* or *Case 2*. The condition depends on the relative marginal benefits of getting the types' common most preferred worker, and on the probability that a firm is the type that also finds the other worker acceptable.

Proposition 3 (The Characterization)**.** *Consider the two-firm, two-worker model.*

(i) If $b_1 > \pi(a)(a_1 - a_2)$, then in equilibrium,

$$\begin{aligned} G_a^*(x) &= \frac{x}{\pi(a)(a_1 - a_2)} \\ &\quad \textit{on the support } \left[0,\, \pi(a)(a_1 - a_2)\right] \\ G_b^*(x) &= \frac{1-\pi(b)}{\pi(b)(b_1 - x)}\left[x - \pi(a)(a_1 - a_2)\right] \\ &\quad \textit{on the support } \left[\pi(a)(a_1 - a_2),\, \pi(b)b_1 + (1-\pi(b))\pi(a)(a_1 - a_2)\right], \end{aligned}$$

regardless of G_c^*, G_d^*. The analogous result holds for types c and d if $d_2 > \pi(c)(c_2 - c_1)$.

(ii) If $b_1 < \pi(a)(a_1 - a_2)$, then in equilibrium,

$$\begin{aligned} G_b^*(x) &= \frac{x(\pi(c)+\pi(d))}{\pi(b)(b_1 - x)} \\ &\quad \textit{on the support } \left[0, \frac{\pi(b)b_1}{1-\pi(a)}\right] \\ G_a^*(x) &= \frac{(1-\pi(a))x - \pi(b)b_1}{\pi(a)(1-\pi(a))(a_1 - a_2)} \\ &\quad \textit{on the support } \left[\frac{\pi(b)b_1}{1-\pi(a)}, \frac{\pi(b)b_1}{1-\pi(a)} + \pi(a)(a_1 - a_2)\right]. \end{aligned}$$

The analogous result holds for types c and d if $d_2 < \pi(c)(c_2 - c_1)$.

Proof. See Appendix. □

Part of the condition in Proposition 3 has a simple intuition. If type b gets a higher value from worker w_1 than the marginal value for type a from getting w_1 instead of w_2, then type b will be willing to pay more than type a for w_1. The role of $\pi(a)$ in the condition is less obvious. Keeping the values fixed, if $\pi(a)$ is relatively low, a type a firm does not need to mix over a large interval to compete against its own type. As a result, if type a firms offer salaries above those offered by type b, there may be an incentive for type b firms to offer salaries higher than the type a firms in order to 'steal' worker w_1 in the event that the other firm is type a. Any such deviation by type b

firms would give type a firms an incentive to lower the support of their distributional strategies to the lower bound of 0.

Proposition 3 also leads to two corollaries. First, all things being equal, the more likely a firm is to face another firm of the same type, the stronger the competitive pressure and the higher the average equilibrium salary offered by that type. Similarly, the more likely a firm is to face another firm with the same most preferred worker, the stronger the competition and the higher the average equilibrium salary offered by the two relevant types. Proofs for both corollaries are provided in the appendix.

Corollary 1. *The higher the probability a firm type has to compete against its own type, the higher (on average) the equilibrium salary offered by that firm type.*

Corollary 2. *For any firm, the higher the probability that the other firm has the same most preferred worker, the higher the equilibrium salary (on average) offered by the firm.*

4 Competition in Large Markets

This section extends the analysis of competitive behavior to larger markets. A general extension, in which the number of distinct workers and therefore the number of firm types grows larger, poses some problems. As the number of workers grows larger, the firm type space grows exponentially, which greatly complicates the equilibrium analysis. Thus, to keep the analysis tractable, we replicate the two-firm, two-worker market. This generates a market with $2n$ firms and $2n$ workers, consisting of n identical $c$lass W_1 workers and n identical $c$lass W_2 workers. We characterize the equilibria for the limit case in which there are a continuum of firms and a continuum of workers, then prove existence, for finite replicated markets, of a Bayesian Nash equilibrium in continuous distributional strategies with interval supports. Moreover, both Proposition 2 and Proposition 3 generalize to replicated markets. Finally, we provide numerical results to show that as the number of replications grows to infinity, the finite market equilibrium strategies converge to the corresponding continuum equilibrium strategies.

4.1 Market Replication

Replication provides a convenient way to conduct a tractable analysis of competitive behavior in large markets. We use, as a baseline, the two-firm, two-worker market, with $F^1 = \{f_1, f_2\}$ and $W = \{w_1, w_2\}$. In an n-replicated market, there are $2n$ firms, $F^n = \{f_1, ..., f_{2n}\}$, along with n copies of w_1, $W_1^n = \{w_1^1, w_1^2, ..., w_1^n\}$, and n copies of w_2, $W_2^n = \{w_2^1, ..., w_2^n\}$. Since w_1^j and w_1^k are identical copies of one another, we assume that all firms are indifferent between any two workers in W_1^n. Likewise, all firms are indifferent between any two workers in W_2^n. As a result, firms' preferences (and from these, their types) are defined as strict orderings over the set $\{W_1, W_2, \emptyset\}$ where W_1 and W_2 are two classes of perfectly substitutable workers.

As in section 2, firm types that prefer being unmatched over every worker are ignored.[5] This leaves four possible firm types that are essentially the same as the types in the two-firm, two-worker model, except that the preferences are over classes of workers W_1 and W_2.

$$
\begin{array}{llllllll}
P_a: & W_1 & W_2 & \emptyset & \qquad P_b: & W_1 & \emptyset & W_2 \\
P_c: & W_2 & W_1 & \emptyset & \qquad P_d: & W_2 & \emptyset & W_1.
\end{array}
$$

Assume that each preference ranking is represented by a pair of values - one for each worker class, W_1 and W_2 - while the value of remaining unmatched is normalized to 0. So, for each type $k \in \{a,b,c,d\}$, $k = (k_1, k_2)$, where k_i is the value of each worker w in the class W_i. For the values to represent the corresponding preference rankings, they must satisfy

$$
\begin{array}{ll}
a_1 > a_2 > 0 & \qquad b_1 > 0 > b_2 \\
c_2 > c_1 > 0 & \qquad d_2 > 0 > d_1.
\end{array}
$$

Each firm knows only its own type, and the types are drawn independently according to the common prior distribution π over $\{a,b,c,d\}$. That is, $\pi(k)$ is the probability that a given firm is a type k firm, or equivalently, has preferences P_k.

4.2 Equilibria in the Continuum Case

Before analyzing the equilibria for a finite replicated market, consider the equilibrium behavior in the limit, when there is a continuum of firms, and continuum of workers. Moreover, suppose that the measure of workers in each class W_1 and W_2 is half the total measure of W. In this environment, since there are infinitely many firms, the aggregate uncertainty about the realized firm types disappears from the market. That is, $\pi(k)$ is the actual proportion, or the measure of type k firms in the market. This is a convenient feature because it makes the equilibrium strategies relatively straightforward functions of the distribution π.

As for the two-firm, two-worker case, the equilibrium strategy for a given type k does not depend on the strategies of the two types k', k'' that have a different most preferred worker class than type k. Thus, as in section 2, when deriving equilibrium strategies, types a and b can be

[5] We may just as well assume that they don't enter the market in the first place.

treated independently from types c and d. Without loss of generality, consider types a and b. The analysis is symmetric for types c and d. Proposition 4 characterizes the equilibria for the limit case. It is broken into two cases based on the relative marginal values of worker class W_1 compared with worker class W_2, for types a and b.

Proposition 4. *The following two cases, (1) and (2), characterize the equilibrium when there are a continuum of firms and a continuum of workers, with two equally large worker classes.*

(1) *Suppose* $b_1 \geq a_1 - a_2$.

(i) *If* $\pi(a) + \pi(b) \leq \frac{1}{2}$, *then* $x_a^* = 0$ *and* $x_b^* = 0$.

(ii) *If* $\pi(a) > \frac{1}{2}$, *then* $x_b^* = a_1 - a_2$ *and*

$$x_a^* = \begin{cases} 0 & \text{with probability } p_a(0) = \frac{2(\pi(a)+\pi(b))-1}{2\pi(a)} \\ a_1 - a_2 & \text{with probability } 1 - p_a(0) \end{cases}.$$

(iii) *If* $\pi(b) > \frac{1}{2}$, *then* $x_a^* = 0$ *and* $x_b^* = b_1$.

(iv) *If* $\pi(a) \leq \frac{1}{2}$, $\pi(b) \leq \frac{1}{2}$, *but* $\pi(a) + \pi(b) > \frac{1}{2}$, *then* $x_b^* = a_1 - a_2$ *and*

$$x_a^* = \begin{cases} 0 & \text{with probability } p_a(0) = \frac{2(\pi(a)+\pi(b))-1}{2\pi(a)} \\ a_1 - a_2 & \text{with probability } 1 - p_a(0) \end{cases}.$$

(2) *Suppose* $b_1 < a_1 - a_2$

(i) *If* $\pi(a) + \pi(b) \leq \frac{1}{2}$, *then* $x_a^* = 0$ *and* $x_b^* = 0$.

(ii) *If* $\pi(a) > \frac{1}{2}$, *then* $x_b^* \in [0, b_1]$ *and*

$$x_a^* = \begin{cases} 0 & \text{with probability } q_a(0) = \frac{2\pi(a)-1}{2\pi(a)} \\ a_1 - a_2 & \text{with probability } 1 - q_a(0) \end{cases}.$$

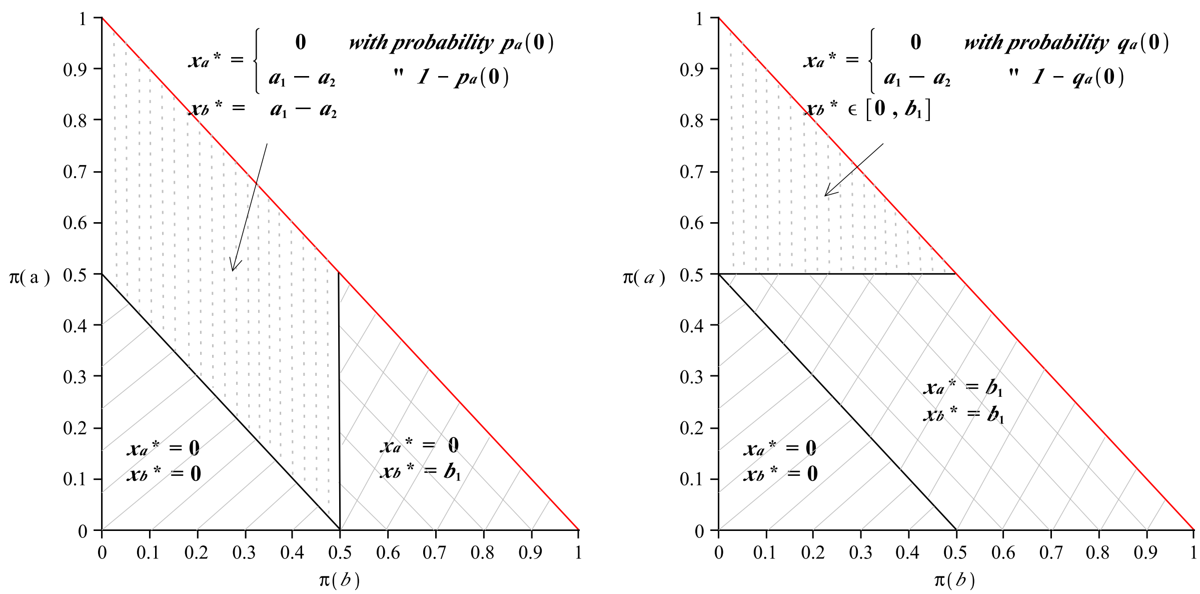
(iii) *If* $\pi(b) > \frac{1}{2}$, *then* $x_a^* = b_1$ *and* $x_b^* = b_1$.

(iv) *If* $\pi(a) \leq \frac{1}{2}$, $\pi(b) \leq \frac{1}{2}$, *but* $\pi(a) + \pi(b) > \frac{1}{2}$, *then* $x_a^* = b_1$ *and* $x_b^* = b_1$.

The formal proof is omitted. Instead, the intuition can be explained with reference to Figures 1a and 1b, which provide graphical illustrations of the two cases in Proposition 4. Each figure plots $\pi(a)$ against $\pi(b)$ and divides the space of probability pairs $(\pi(b), \pi(a))$ into segments for each subcase of the equilibrium characterization. In both Figure 1a and Figure 1b, the bottom left triangle corresponds to the case in which there is an excess supply of class W_1 workers, and therefore no competition between types a and b. In this case, both types can choose a salary equal

Figure 1: Continuum Equilibria for Firm Types a and b in Cases (1) and (2) from Proposition 4

(a) Case (1): $b_1 \geq a_1 - a_2$ (b) Case (2): $b_1 < a_1 - a_2$



to zero and be assured of matching with a worker from W_1. Thus, $x_a^* = x_b^* = 0$ for both cases when $\pi(a) + \pi(b) \leq \frac{1}{2}$.

Figure 1a merges the subcase in which $\pi(a) > \frac{1}{2}$ with the subcase in which $\pi(a) \leq \frac{1}{2}$ and $\pi(b) \leq \frac{1}{2}$, but $\pi(a) + \pi(b) > \frac{1}{2}$, since in each, type a firms mix between 0 and $a_1 - a_2$ with probability $p_a(0) = \frac{2[\pi(a)+\pi(b)]-1}{2\pi(a)}$, while type b firms choose $a_1 - a_2$. In each of these subcases, the competitive pressure among Finally, in the case when $\pi(b) > \frac{1}{2}$, type b firms compete with each other and push the salary up to their marginal value from a class W_1 worker, while type a firms know that they will not be matched with a class W_1 worker and so choose a salary of 0.

In Figure 1b, we can likewise merge the subcase in which $\pi(b) > \frac{1}{2}$ with the subcase in which $\pi(a) \leq \frac{1}{2}$ and $\pi(b) \leq \frac{1}{2}$, but $\pi(a) + \pi(b) > \frac{1}{2}$, since in each subcase, both type a firms and type b firms choose a salary of b_1. When $\pi(a) > \frac{1}{2}$, type a firms mix between 0 and $a_1 - a_2$ with probability $q_a(0) = \frac{2\pi(a)-1}{2\pi(a)}$, while type b firms choose a salary in the interval $[0, b_1]$. This is because type a firms drive the salary for a class W_1 worker up to $a_1 - a_2 > b_1$, so that type b firms are never matched with anyone. Since some of the type a firms will miss out on a class W_1 worker, they mix between the salary $a_1 - a_2$ and 0.

4.3 Finite Replicated Markets

As in both the two-firm, two-worker and the continuum cases, we prove existence of a unique Bayesian Nash equilibrium in continuous distributional strategies on interval support, for given parameters of the model. The full characterization must be solved for implicitly using the indifference equations derived in the proof, which is given in the Appendix. Most importantly, we find that the separation result obtained in Proposition 2 extends to the equilibrium for any finite replicated market, with respect to firm types with a common most preferred worker *class*.

Proposition 5. *Given any finite replicated market with $2n$ firms, n workers in class W_1 and n workers in class W_2, there exists an equilibrium $(G_a^*(\cdot), G_b^*(\cdot), G_c^*(\cdot), G_d^*(\cdot))$ such that $G_k^*(\cdot)$ is a continuous distribution with interval support in the salary space, for all $k = a, b, c, d$. The equilibrium supports for types a and b satisfy*

$$\begin{aligned} & 0 = \underline{x}_a < \overline{x}_a = \underline{x}_b < \overline{x}_b \\ \text{or} \quad & 0 = \underline{x}_b < \overline{x}_b = \underline{x}_a < \overline{x}_a. \end{aligned}$$

The analogous result holds for the equilibrium supports of types c and d.

Proof. See Appendix. □

5 Convergence of Finite Market Equilibria

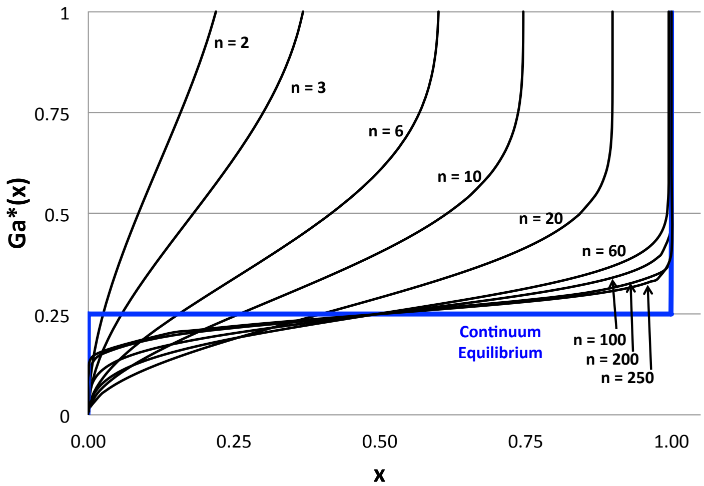
This section shows numerically that the replicated market equilibrium strategies converge to the corresponding continuum equilibrium as the number of replications goes to infinity. The convergence is illustrated by simulating replicated markets for the market presented in Example 1. Recall that $a = (2,1)$, $b = (2,-1)$, $c = (1,2)$, and $d = (-1,2)$, while $\pi(a) = \frac{1}{2}$, $\pi(b) = \frac{1}{8}$, $\pi(c) = \frac{1}{4}$, and $\pi(d) = \frac{1}{8}$. The corresponding continuum equilibrium is as follows,

$$\begin{aligned} x_a^* &= \begin{cases} 0 & \text{with probability } \frac{1}{4} \\ 1 & \text{with probability } \frac{3}{4} \end{cases} \\ x_b^* &= 1, \qquad x_c^* = 0, \qquad x_d^* = 0. \end{aligned}$$

The equilibrium distribution for a type a firm in an n-replicated market satisfies

$$\begin{aligned} \underline{x}_a &= 0 \\ \overline{x}_a &= (a_1 - a_2)\left[\sum_{j=0}^{n-1} \sum_{k=n+1-j}^{2n-1-j} \frac{(2n-1)!\pi(b)^j \pi(a)^k [1-\pi(b)-\pi(a)]^{2n-1-j-k}}{j!k!(2n-1-j-k)!}\right] \end{aligned}$$

Figure 2: Replicated Market Equilibria for Example 1. Type a firms' equilibrium strategies for $n \in \{2,3,6,10,20,60,100,200,250\}$.



and for all $x \in [\underline{x}_a, \overline{x}_a]$

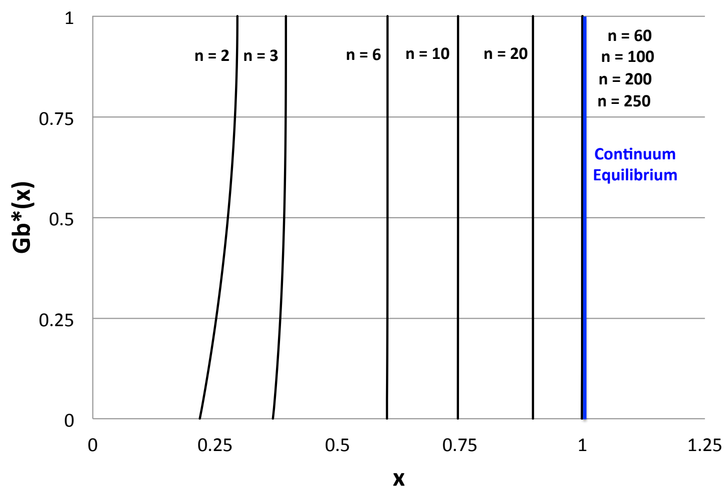
$$x = (a_1 - a_2) \sum_{j=0}^{n-1} \sum_{k=n+1-j}^{2n-1-j} \left[\frac{(2n-1)!\pi(b)^j \pi(a)^k [1-\pi(a)-\pi(b)]^{2n-1-j-k}}{j!k!(2n-1-j-k)!} \times \sum_{t=0}^{n-j-1} \binom{k}{t} G_a^*(x)^{k-t} [1-G_a^*(x)]^t \right].$$

The last equation can be solved, given any value of $G_a^*(x)$, for the corresponding value of x. Then the pairs $(x, G_a^*(x))$ that satisfy the indifference equations for a given value of n can be used to trace out the equilibrium distribution for a type a firm in a market that has been replicated n times. Figure 2 plots these pairs $(x, G_a^*(x))$ with x on the horizontal axis and $G_a^*(x)$ on the vertical axis for several different values of n.

For smaller sized markets, $n \in \{2,3,6,10,20\}$, increasing the market size shifts more density to higher salaries and expands the equilibrium support. However, after n grows large enough (for example, by $n = 60$), the equilibrium support approaches its upper bound of 1 (the marginal value for a class W_1 worker, $a_1 - a_2$). Then for any larger replicated markets, type a firms shift greater weight towards salaries very close to the upper bound. In order to maintain indifference over the support, they must also shift density to the very low salaries (close to 0), which leads to a sequence of CDFs that approaches the continuum equilibrium as n approaches infinity.

The same procedure can be run for type b firms, and also for types c and d. For type b firms, the corresponding equilibrium support has a lower bound equal to the upper bound of the support for type a firms, by the separation result. The size of the support for a type b firm's equilibrium

Figure 3: Replicated Market Equilibria for Example 1. Type b firms' equilibrium strategies for $n \in \{2,3,6,10,20,60,100,200,250\}$.



strategy is decreasing with the size of the market, since the probability of having to compete with n other type b firms goes to zero. Then, since the lower bound for sufficiently large n is equal to 1, the equilibrium strategies converge towards the continuum equilibrium strategy, which places the entire mass on a salary equal to 1. Figure 3 shows the simulated calculations for equilibrium strategies of a type b firm in different sized markets. Of particular interest is the observation that by $n = 6$, there is already almost no competitive pressure for a type b firm to compete against another type b firm. Even with so few replications, a type b firm realizes that the chances of there being more than 5 other type b firms (among the other 11 firms) is almost zero. As a result, type b firms choose a salary just high enough to ensure that they will be ranked higher by the workers than any type a firms. Similarly, as the number of replications increases, for each type c and type d firm, the probability that there are $n-1$ other type c or type d firms in the market goes to zero, removing any competitive pressure in their pursuit of a w_2 worker. Thus, the equilibrium strategies converge to the continuum equilibrium strategies, where type c and type d firms choose a salary of zero. Thus, as the market is replicated, competitive pressure is enhanced only for the most popular worker class. However, even then, there is no pressure to compete for the firm type with a higher relative marginal value for the popular workers.

6 Conclusion

In this article, we study the competitive behavior of firms in a unique labor market where firms must commit to offer the same salary to any worker they wish to hire. The firms do not know the preferences of other firms, and so face an uncertain level of competitive pressure. In equilibrium, we find that competition is *localized*, in the sense that the different firm types mix over adjacent, but non-overlapping salary intervals. The characteristics of competition are robust to the size of the market, but as it grows, the uncertainty disappears until there is competition only for the most popular class of workers.

The article presents several key results for both the two-firm, two-worker model, and for larger, replicated markets. First, we show that, in the two-firm, two-worker case, there does not exist a Bayesian Nash equilibrium in pure strategies. We then proceed to a characterization of the unique equilibrium in distributional strategies with interval support. This characterization shows that strategy supports for types with a common most preferred worker are adjacent, such that the type with a higher relative marginal value for the preferred worker pays a premium to ensure it is selected before any firm of the other type.

We extend the analysis to larger markets by replicating the baseline two-firm, two-worker model. In the limit, when there are a continuum of firms and a continuum of workers, there is no aggregate uncertainty about the realization of firm types. As a result, competition in equilibrium is confined to the class of workers that are more popular, while the salaries of other workers fall to zero. For finitely replicated markets, we prove the existence of distributional Bayesian Nash equilibrium strategies that exhibit the same separation result obtained for the two-firm two-worker case in the equilibrium supports, for types with a common most preferred worker class. Finally, we report numerical results to show that, as the number of replications grow, the equilibrium strategies in the finite replicated market approach the corresponding continuum equilibrium strategies. That is, when markets become larger, aggregate uncertainty about the actual types of other firms dissipates and reduces the level of competitive pressure on salaries.

A natural extension of this paper is to consider a more general n-firm, n-worker model. In this article, large markets are generated by replicating the two-firm, two-worker model, which controls the size of the type space and keeps the analysis tractable. A limitation of this approach is that all workers in the same class are treated as identical from the perspective of the firms. Future work might focus on relaxing this assumption, while maintaining tractability. An alternative approach may be to relax the restriction that firms with the same preference ordering must have the same valuation or utility representation for that ordering. For instance, suppose each firm's type is a pair of values $\theta = (x,y)$, each drawn independently from some interval $[\underline{\theta},\overline{\theta}]$ according to a given distribution.

Another potential extension is to introduce some correlation structure into the beliefs about other firms' types. Since in many cases, firms will be looking for candidates with similar characteristics, it may be more realistic to relax the assumption that types are drawn independently from the common prior. Such an extension will likely increase the degree of perceived competitive pressure for the more common firm types, leading to higher equilibrium salaries. Finally, we might also consider allowing for more firms than workers, so as to intensify the baseline level of competitive pressure for all workers.

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Appendix

Proof of Proposition 1. Consider any arbitrary pair of strategies $(\mathbf{s}_1, \mathbf{s}_2)$ and suppose firm 1's type is a. Notice that, if firm 2's type is either of c or d, then regardless of $s_1(a)$, firm 1 is matched with worker w_1. The case that matters is when firm 2's type is a or b, because then the outcome depends on the salaries announced by the firms. If firm 2 is playing $\mathbf{s}_2$, then firm 1's best response is to announce $s_1(a) = max\{s_2(a), s_2(b)\} + \varepsilon$ as long as $s_1(a) \leq a_1 - a_2$. If $max\{s_2(a), s_2(b)\} \geq a_1 - a_2$, then firm 1's best response is to announce $s_1(a) = 0$. However, given the choice of firm 1, $s_1(a) = max\{s_2(a), s_2(b)\} + \varepsilon$, firm 2's best response, if it is type a, is to offer $s_2(a) = s_1(a) + \varepsilon$, up to $s_2(a) \leq a_1 - a_2$. The same type of incremental best responses exist for type b firms, and by symmetry, also for types c and d. Since the probability distribution over types has full support (i.e. no type occurs with zero probability), every type faces some chance of being drawn into this cycle.

Thus, firms who have a common most preferred worker will continue to outbid each other until the marginal benefit of 'winning' the worker is equal to the marginal benefit of not winning (either 0 or the marginal benefit of the other worker). However, once that point is reached, the best response is to announce a salary of 0, and the bidding-up process will begin all over again. □

Proof of Lemma 1. Let $[\underline{x}_a, \bar{x}_a]$ and $[\underline{x}_b, \bar{x}_b]$ be the equilibrium supports for types a and b respectively. Suppose by means of contradiction that neither $\underline{x}_a$ nor $\underline{x}_b$ is equal to 0. Consider $0 < \underline{x}_a \leq \underline{x}_b$. Type a's expected payoff from $x = \underline{x}_a$ is

$$\mathbb{E}U_a(\underline{x}_a) = [\pi(a) + \pi(b)]a_2 + [\pi(c) + \pi(d)]a_1 - \underline{x}_a$$

and for any $x \in [0, \underline{x}_a)$, type a's expected payoff is

$$\begin{aligned} \mathbb{E}U_a(x) &= [\pi(a) + \pi(b)]a_2 + [\pi(c) + \pi(d)]a_1 - x \\ &< \mathbb{E}U_a(\underline{x}_a). \end{aligned}$$

This means that $[\underline{x}_a, \bar{x}_a]$ cannot be an equilibrium support unless $\underline{x}_a = 0$ or $0 \leq \underline{x}_b < \underline{x}_a$.

If $0 < \underline{x}_b \leq \underline{x}_a$, type b's expected payoff from $x = \underline{x}_b$ is

$$\mathbb{E}U_b(\underline{x}_b) = (b_1 - \underline{x}_b)[\pi(c) + \pi(d)].$$

That is, at the lower bound of type b's equilibrium support, a firm of type b does not get matched to a worker unless the other firm is type c or type d. But in those cases, the salary does not affect the outcome, so that choosing a salary of $\underline{x}_b > 0$ is strictly dominated by $x = 0$. Thus, $[\underline{x}_b, \bar{x}_b]$ cannot be an equilibrium support unless $\underline{x}_b = 0$ or $0 \leq \underline{x}_a < \underline{x}_b$. Therefore, in equilibrium, we must have either $\underline{x}_a = 0$ or $\underline{x}_b = 0$. □

Proof of Lemma 2. Suppose $\bar{x}_a < \underline{x}_b$. Then $\forall x \in (\bar{x}_a, \underline{x}_b)$, type b's expected payoff is

$$\begin{aligned} \mathbb{E}U_b(x) &= (b_1 - x)(1 - \pi(b)) \\ &> (b_1 - \underline{x}_b)(1 - \pi(b)) = \mathbb{E}U_b(\underline{x}_b), \end{aligned}$$

contradicting the inclusion of $\underline{x}_b$ in the equilibrium support for type b. The proof is similar for the case when $\bar{x}_b < \underline{x}_a$. Since the supports are intervals by assumption, there are no other cases to be considered. □

Proof of Proposition 2. <u>*Case 1:*</u> $\quad 0 = \underline{x}_a < \underline{x}_b \leq \bar{x}_a < \bar{x}_b$

Consider the interval $[\underline{x}_b, \bar{x}_a]$, on which both firm types a and b make offers. Suppose by means of contradiction that this interval has a nonempty interior. For type a, the expected payoff for any salary in the interval is

$$\mathbb{E}U_a(x) = (a_1 - a_2)\Big[\pi(a)G_a^*(x) + \pi(b)G_b^*(x)\Big] - (a_1 - a_2)\Big[\pi(a) + \pi(b)\Big] + a_1 - x$$

for all $x \in [\underline{x}_b, \bar{x}_a]$. To make type a indifferent on the interval, we must have

$$g_a^*(x) = \frac{1}{\pi(a)(a_1 - a_2)} - \frac{\pi(b)}{\pi(a)} g_b^*(x), \qquad \forall x \in [\underline{x}_b, \bar{x}_a]. \tag{A.1}$$

Integrating equation (A.1) with respect to x yields

$$G_a^*(x) = G_a^*(\underline{x}_b) + \frac{x - \underline{x}_b}{\pi(a)(a_1 - a_2)} - \frac{\pi(b)}{\pi(a)} G_b^*(x), \tag{A.2}$$

for all $x \in [\underline{x}_b, \bar{x}_a]$.

On the other hand, type b has expected payoff

$$\mathbb{E}U_b(x) = (b_1 - x)\Big[\pi(a)G_a^*(x) + \pi(b)G_b^*(x) + \pi(c) + \pi(d)\Big], \qquad \forall x \in [\underline{x}_b, \bar{x}_a].$$

For type b to be indifferent on the interval, we need

$$g_b^*(x) - \frac{G_b^*(x)}{b_1 - x} = \frac{\pi(c) + \pi(d) + \pi(a)G_a^*(x)}{\pi(b)(b_1 - x)} - \frac{\pi(a)}{\pi(b)} g_a^*(x).$$

Solving this differential equation and using integration by parts, we find

$$G_b^*(x) = \frac{\eta(x) - \eta(\underline{x}_b)}{\pi(b)(b_1 - x)}, \qquad \forall x \in [\underline{x}_b, \bar{x}_a], \tag{A.3}$$

where $\eta(s) = s(\pi(c)+\pi(d)) - \pi(a)G_a^*(s)(b_1 - s)$. Then, by substituting equation (A.3) into equation (A.2) and simplifying, we must have

$$G_a^*(\underline{x}_b) = \frac{1}{\pi(a)}\left[\frac{b_1 - x}{a_1 - a_2} - \pi(c) - \pi(d)\right] \tag{A.4}$$

for **every** $x \in [\underline{x}_b, \bar{x}_a]$. Since $\pi(a) > 0$ and $a_1 > a_2$, the right hand side of equation (A.4) is strictly decreasing in x, which implies that the interior of the interval $[\underline{x}_b, \bar{x}_a]$ must be empty. That is, $\underline{x}_b = \bar{x}_a$.

<u>*Case 2:*</u> $\quad 0 = \underline{x}_b < \underline{x}_a \leq \bar{x}_b < \bar{x}_a$

Consider the interval $[\underline{x}_a, \bar{x}_b]$, on which both type a and type b make offers. Suppose again by contradiction that this interval has a nonempty interior. For type b, the expected payoff for any salary in the interval is

$$(b_1 - x)\Big[\pi(a)G_a^*(x) + \pi(b)G_b^*(x) + \pi(c) + \pi(d)\Big].$$

To make type b indifferent on the interval, we must have

$$g_b^*(x) - \frac{G_b^*(x)}{b_1 - x} = \frac{\pi(c)+\pi(d)+\pi(a)G_a^*(x)}{\pi(b)(b_1 - x)} - \frac{\pi(a)}{\pi(b)}g_a^*(x). \tag{A.5}$$

Solving the differential equation (A.5) and using integration by parts to simplify the solution, we have

$$G_b^*(x) = \frac{(x - \underline{x}_a)(\pi(c)+\pi(d))}{\pi(b)(b_1 - x)} - \frac{\pi(a)}{\pi(b)}G_a^*(x) + \frac{k}{b_1 - x}, \qquad \forall x \in [\underline{x}_a, \bar{x}_b], \tag{A.6}$$

where k is some constant of integration. Using the fact that $G_b^*(\bar{x}_b) = 1$, we can solve for

$$k = b_1 - \bar{x}_b + \frac{\pi(a)}{\pi(b)}G_a^*(\bar{x}_b)(b_1 - \bar{x}_b) - \frac{\pi(c)+\pi(d)}{\pi(b)}(\bar{x}_b - \underline{x}_a). \tag{A.7}$$

Substituting (A.7) into (A.6) and simplifying gives

$$G_b^*(x) = \frac{b_1 - \bar{x}_b}{\pi(b)(b_1 - x)}\Big[\pi(b) + \pi(a)G_a^*(\bar{x}_b)\Big] - \frac{\pi(a)}{\pi(b)}G_a^*(x) - \frac{\pi(c)+\pi(d)}{\pi(b)(b_1 - x)}(\bar{x}_b - x). \tag{A.8}$$

For type a, the expected payoff for any salary in the interval is

$$\mathbb{E}U_a(x) = (a_1 - a_2)\Big[\pi(a)G_a^*(x) + \pi(b)G_b^*(x)\Big] - (a_1 - a_2)\Big[\pi(a) + \pi(b)\Big] + a_1 - x,$$

for all $x \in [\underline{x}_a, \bar{x}_b]$. Solving to make type a indifferent and using the fact that $G_a^*(\underline{x}_a) = 0$ by assumption, we have

$$G_a^*(x) = \frac{x - \underline{x}_a}{\pi(a)(a_1 - a_2)} - \frac{\pi(b)}{\pi(a)}\left[G_b^*(x) - G_b^*(\underline{x}_a)\right]. \tag{A.9}$$

Substituting equation (A.8) into equation (A.9) and simplifying yields

$$\frac{x - \underline{x}_a}{a_1 - a_2} + \frac{[\pi(c) + \pi(d)](\bar{x}_b - x)}{b_1 - x} - \frac{b_1 - \bar{x}_b}{b_1 - x}(\pi(b) + \pi(a)G_a^*(\bar{x}_b) \tag{A.10}$$

$$= \frac{[\pi(c) + \pi(d)](\bar{x}_b - \underline{x}_a)}{b_1 - \underline{x}_a} - \frac{b_1 - \bar{x}_b}{b_1 - \underline{x}_a}(\pi(b) + \pi(a)G_a^*(\bar{x}_b)) \tag{A.11}$$

for all $x \in [\underline{x}_a, \bar{x}_b]$. Notice that the right hand side of equation (A.10) is constant. To maintain equality, the derivative of the left hand side with respect to x must be 0 for **every** $x \in [\underline{x}_a, \bar{x}_b]$. However, this derivative,

$$\frac{1}{a_1 - a_2} - \frac{b_1 - \bar{x}_b}{(b_1 - x)^2}(\pi(c) + \pi(d) - \pi(b) - \pi(a)G_a^*(\bar{x}_b)), \tag{A.12}$$

changes with x unless $b_1 - \bar{x}_b = 0$ or $\pi(c) + \pi(d) = \pi(a)G_a^*(\bar{x}_b) + \pi(b)$. Furthermore, in those cases, the equation (A.12) equals $\frac{1}{a_1 - a_2} > 0$ since $a_1 > a_2$. It follows then that the interior of $[\underline{x}_a, \bar{x}_b]$ must be empty; that is, $\underline{x}_a = \bar{x}_b$.

Case 3: $\quad 0 = \underline{x}_b \le \underline{x}_a < \bar{x}_a \le \bar{x}_b$

Consider the interval $[\underline{x}_a, \bar{x}_a]$, on which both types make offers. As above, we can solve for

$$G_b^*(x) = \frac{\pi(c) + \pi(d)}{\pi(b)(b_1 - x)}(x - \underline{x}_a) - \frac{\pi(a)}{\pi(b)}G_a^*(x), \tag{A.13}$$

for all $x \in [\underline{x}_a, \bar{x}_a]$, and

$$G_a^*(x) = \frac{x - \underline{x}_a}{\pi(a)(a_1 - a_2)} - \frac{\pi(b)}{\pi(a)}\left[G_b^*(x) - G_b^*(\underline{x}_a)\right], \tag{A.14}$$

for all $x \in [\underline{x}_a, \bar{x}_a]$. In order to satisfy both equation (A.13) and equation (A.14), we substitute the latter into the former and simplify, to obtain

$$G_b^*(\underline{x}_a) = \left[\frac{\pi(c) + \pi(d)}{\pi(b)(b_1 - x)} - \frac{1}{b_1(a_1 - a_2)}\right](x - \underline{x}_a). \tag{A.15}$$

The left hand side of equation (A.15) is a constant. To maintain the equality, we must have the derivative of the right hand side equal to 0 for **every** $x \in [\underline{x}_a, \overline{x}_a]$, which means

$$\left[\frac{\pi(c)+\pi(d)}{\pi(b)(b_1-x)} - \frac{1}{b_1(a_1-a_2)}\right] = \frac{\pi(c)+\pi(d)}{\pi(b)(b_1-x)^2}(x-\underline{x}_a) \tag{A.16}$$

However, equation (A.16) has a unique solution, which implies that $\underline{x}_a = \overline{x}_a$. Since we have already ruled out pure strategies as best responses, it follows that there are no equilibria of the form described by *Case 3*.

Case 4: $\quad 0 = \underline{x}_a \leq \underline{x}_b < \overline{x}_b \leq \overline{x}_a$

The steps to prove that *Case 4* cannot occur are analogous to the steps for *Case 3*. We conclude that in order to simultaneously satisfy the conditions for indifference by both types on the interval $[\underline{x}_b, \overline{x}_b]$, we must have $\underline{x}_b = \overline{x}_b$, which we know cannot be true, since we have already ruled out pure strategies. Thus, there are no equilibria of the form described by *Case 4*. □

Proof of Proposition 3. By Proposition 2, in any Bayesian Nash equilibrium, the supports for the strategies of two types with a common most preferred worker (say, a and b) must satisfy either *Case 1* with $\overline{x}_a = \underline{x}_b$, or *Case 2* with $\overline{x}_b = \underline{x}_a$.

Case 1:

Consider *Case 1* and suppose that $G_a^*(\cdot)$ and $G_b^*(\cdot)$ are the candidate equilibrium strategies for types a and b. For each salary x in the interval $[0, \overline{x}_a]$, type a firms have an expected payoff equal to

$$\mathbb{E}U_a(x) = a_1\pi(a)G_a^*(x) + a_2\pi(a)(1-G_a^*(x)) + a_2\pi(b)] + a_1[\pi(c)+\pi(d)] - x.$$

Since the firm needs to be indifferent between any salary that is offered as part of its equilibrium strategy, we must have

$$g_a^*(x) = \frac{1}{\pi(a)(a_1-a_2)} \qquad \forall x \in (0, \overline{x}_a]. \tag{A.17}$$

Integrating with respect to x yields

$$\begin{aligned} G_a^*(x) &= G_a^*(0) + \int_0^x \frac{1}{\pi(a)(a_1-a_2)} ds && \text{(A.18)} \\ &= G_a^*(0) + \frac{x}{\pi(a)(a_1-a_2)}, && \text{(A.19)} \end{aligned}$$

for all $x \in [0, \overline{x}_a]$. We assume that when both firms choose a salary of 0, the workers flip a coin if they have to decide between the two offers. As a result, the payoff from $x = 0$ is strictly less than from some small $\varepsilon > 0$. Thus, $G_a^*(0) = 0$. Then we have

$$G_a^*(x) = \frac{x}{\pi(a)(a_1 - a_2)} \qquad \forall x \in [0, \bar{x}_a], \tag{A.20}$$

and since $G_a^*(\bar{x}_a) = 1$, we can solve for $\bar{x}_a = \pi(a)(a_1 - a_2)$.

Similarly, for each salary x in the interval $[\bar{x}_a, \bar{x}_b]$, type b firms have an expected payoff equal to

$$\mathbb{E}U_b(x) = (b_1 - x)[\pi(b)G_b^*(x) + 1 - \pi(b)].. \tag{A.21}$$

In order for type b firms to be indifferent between all the salaries in the interval $[\bar{x}_a, \bar{x}_b]$, we must have

$$g_b^*(x) - \frac{G_b^*(x)}{b_1 - x} = \frac{1 - \pi(b)}{\pi(b)(b_1 - x)}, \qquad \forall x \in (\bar{x}_a, \bar{x}_b] \tag{A.22}$$

Solving the differential equation in A.22 gives

$$G_b^*(x) = \frac{1 - \pi(b)}{\pi(b)(b_1 - x)}(x - \bar{x}_a) + \frac{c}{b_1 - x}, \qquad \forall x \in [\bar{x}_a, \bar{x}_b]. \tag{A.23}$$

$G_b^*(\bar{x}_b) = 1$ allows us to solve for

$$c = b_1 - \bar{x}_b - \frac{1 - \pi(b)}{\pi(b)}(\bar{x}_b - \bar{x}_a),$$

and substitue into equation A.23, which simplifies then to

$$G_b^*(x) = 1 - \frac{\bar{x}_b - x}{\pi(b)(b_1 - x)}, \qquad \forall x \in [\bar{x}_a, \bar{x}_b]. \tag{A.24}$$

Having solved for $\bar{x}_a = \pi(a)(a_1 - a_2)$, we use the fact that $G_b^*(\bar{x}_a) = 0$ to solve for

$$\bar{x}_b = \pi(b)b_1 + (1 - \pi(b))\pi(a)(a_1 - a_2). \tag{A.25}$$

Substituting equation A.25 into equation A.24 and simplifying gives the equilibrium strategy for type b firms,

$$\begin{aligned} G_b^*(x) &= \frac{1 - \pi(b)}{\pi(b)(b_1 - x)}[x - \pi(a)(a_1 - a_2)] \\ &\quad \text{on the support } \Big[\pi(a)(a_1 - a_2),\, \pi(b)b_1 + (1 - \pi(b))\pi(a)(a_1 - a_2)\Big] \end{aligned}$$

The condition that $b_1 > \pi(a)(a_1 - a_2)$ follows immediately, since if $b_1 < \pi(a)(a_1 - a_2)$, then $\left[\pi(a)(a_1 - a_2),\, \pi(b)b_1 + (1-\pi(b))\pi(a)(a_1 - a_2)\right]$ is not an interval; the upper bound is less than the lower bound. This takes care of *Case 1*.

Case 2:

Now consider *Case 2*. For each salary x in the interval $[0, \bar{x}_b]$, type b firms have an expected payoff equal to

$$\mathbb{E}U_b(x) = (b_1 - x)[\pi(b)G_b^*(x) + \pi(c) + \pi(d)]. \tag{A.26}$$

In order for type b firms to be indifferent between all the salaries in the interval $[0, \bar{x}_b]$, we must have

$$g_b^*(x) - \frac{G_b^*(x)}{b_1 - x} = \frac{\pi(c) + \pi(d)}{\pi(b)(b_1 - x)}, \qquad \forall x \in (0, \bar{x}_b]. \tag{A.27}$$

Solving the differential equation in A.27 gives

$$G_b^*(x) = \frac{\pi(c) + \pi(d)}{\pi(b)(b_1 - x)}x + \frac{c}{b_1 - x}, \qquad \forall x \in (0, \bar{x}_b]. \tag{A.28}$$

For the same reasons as above, we can easily verify that $G_b^*(0) = 0$, which implies $c = 0$, and therefore

$$G_b^*(x) = \frac{\pi(c) + \pi(d)}{\pi(b)(b_1 - x)}x \qquad \forall x \in [0, \bar{x}_b]. \tag{A.29}$$

Since $G_b^*(\bar{x}_b) = 1$, we can solve for $\bar{x}_b = \frac{\pi(b)b_1}{1-\pi(a)}$.

For a type a firm, the expected payoff for each salary $x \in [\bar{x}_b, \bar{x}_a]$ is given by

$$\mathbb{E}U_a(x) = a_1\pi(a)G_a^*(x) + a_2\pi(a)(1 - G_a^*(x)) + a_1(1 - \pi(a)) - x. \tag{A.30}$$

For firm type a to be indifferent on the interval, we must have

$$g_a^*(x) = \frac{1}{\pi(a)(a_1 - a_2)}, \qquad \forall x \in [\bar{x}_b, \bar{x}_a]. \tag{A.31}$$

Integrating and using the fact that $G_a^*(\bar{x}_b) = 0$, we obtain

$$G_a^*(x) = \frac{x - \bar{x}_b}{\pi(a)(a_1 - a_2)}, \tag{A.32}$$

for all $x \in [\bar{x}_b, \bar{x}_a]$. Then substituting $\bar{x}_b = \frac{\pi(b)b_1}{1-\pi(a)}$ into equation A.32 and simplifying gives the equilibrium strategy for type a firms,

$$G_a^*(x) = \frac{(1-\pi(a))x - \pi(b)b_1}{\pi(a)(1-\pi(a))(a_1-a_2)} \tag{A.33}$$

$$\text{on the support } \left[\frac{\pi(b)b_1}{1-\pi(a)}, \bar{x}_a\right]. \tag{A.34}$$

Finally, using $G_a^*(\bar{x}_a) = 1$ allows us to solve for $\bar{x}_a = \frac{\pi(b)b_1}{1-\pi(a)} + \pi(a)(a_1-a_2)$. This takes care of *Case 2*. The proof for types c and d is identical, except for the notation. □

Proof of Corollary 1. In *Case 1*, the expected salary offer of firm type a is just the expected value of a uniform random variable on $\left[0, \pi(a)(a_1-a_2)\right]$ - that is,

$$\mathbb{E}(x_a) = \frac{\pi(a)(a_1-a_2)}{2},$$

which is strictly increasing in $\pi(a)$. For type b, the expected salary is

$$\mathbb{E}(x_b) = \int_{\pi(a)(a_1-a_2)}^{\pi(b)b_1+(1-\pi(b))\pi(a)(a_1-a_2)} x \cdot g_b^*(x)dx. \tag{A.35}$$

Recall that the distribution

$$G_b^*(x) = \frac{1-\pi(b)}{\pi(b)(b_1-x)}\left[x - \pi(a)(a_1-a_2)\right],$$

which gives

$$g_b^*(x) = \frac{1-\pi(b)}{\pi(b)}\left(\frac{b_1 - \pi(a)(a_1-a_2)}{(b_1-x)^2}\right). \tag{A.36}$$

Since $b_1 > \pi(a)(a_1-a_2)$ for this case, we have $b_1 > x$, and $g_b^*(x)$ is increasing in x.

Furthermore,

$$\frac{\partial g_b^*(x)}{\partial \pi(b)} = -\frac{b_1 - \pi(a)(a_1-a_2)}{[\pi(b)(b_1-x)]^2} < 0 \tag{A.37}$$

implies that $g_b^*(x)$ decreases as $\pi(b)$ increases. However, $\frac{\partial^2 g_b^*(x)}{\partial \pi(b)^2}$ is also negative, which means that the decrease in $g_b^*(x)$ from an increase in $\pi(b)$ is more severe for lower values of x.

Since the upper bound of the integration is increasing in $\pi(b)$, the expected value of the salary offered by type b must be increasing with $\pi(b)$, since we assign positive weight to higher salaries not previously included, and the weight attached to those salaries that were previously included falls more for lower salaries than higher salaries.

The proof for *Case 2* uses a similar series of calculations to verify that the expected salary is increasing in the probability of the firm type. It should also be mentioned that, for some values of b_1 and $a_1 - a_2$, as $\pi(a)$ is increasing, it may cause the equilibrium to switch from *Case 1* to *Case 2*. In this case, there is some chance that the average equilibrium salary will jump down (or up), however, for any further increases in $\pi(a)$, the result will continue to hold. □

Proof of Corollary 2. First, we show that the average equilibrium salary for a given firm type is non-decreasing in the probability of the other firm type with the same most preferred worker. Together with Corollary 1, this implies the result. For *Case 1*, type a's expected salary does not depend on $\pi(b)$. On the other hand, for type b, both the lower and upper bounds of the support increase with $\pi(a)$. Futhermore, $g_b^*(x)$ is decreasing in $\pi(a)$, but does so more severely for lower salaries. Therefore, the expected salary for type b increases with $\pi(a)$.

Again, we follow the same steps for proving the result in *Case 2*, and show that in that case, the expected salary for type a is actually increasing in $\pi(b)$. Also, as discussed in the proof for Corollary 1, for some values of b_1 and $a_1 - a_2$, an increase in $\pi(a)$ may cause the equilibrium to switch from *Case 1* to *Case 2*. For type b, this means that the expected salary ought to jump down discretely, lowering the expected salaries of type b for high enough values of $\pi(a)$. Nevertheless, within a particular case, the expected salary for type b is increasing in $\pi(a)$. □

Proof of Proposition 5. Let $\pi(a) + \pi(b) \geq \frac{1}{2}$ and suppose $b_1 \geq a_1 - a_2$. We conjecture the existence of a pair of equilibrium distributions $(G_a^*(\cdot), G_b^*(\cdot))$, with supports $[0, \bar{x}_a]$ and $[\bar{x}_a, \bar{x}_b]$, respectively. In order to prove that these are in fact equilibrium strategies, we first need to show indifference between each of the salaries in their corresponding equilibrium supports.

Consider type a firms. For any type a firm f, the expected utility of a salary $x_f \in (0, \bar{x}_a)$ is

$$\mathbb{E}U_a(x_f) = (a_1 - a_2) \cdot \mathbb{P}r[\mu(f) \in W_1] + a_2 - x_f. \tag{A.38}$$

The probability $\mathbb{P}r[\mu(f) \in W_1]$ consists of two terms that capture, respectively,

(1) the probability that the actual number of type a firms and type b firms is less than or equal to n (the number of class W_1 workers), plus
(2) the probability that
    - the actual number of type a's and type b's is greater than n,
    - the number of type b's is less than n, and

– the number of type a firms that choose $x > x_f$ is less than or equal to $n-$ the number of type b's.

In any other realization of types and salaries, the firm f is matched with a worker $w \in W_2$ and so receives a payoff of a_2.

If we let j denote the number of type b firms and k denote the number of type a firms out of the $2n-1$ other firms, then we can rewrite the first term of $\mathbb{P}r[\mu(f) \in W_1]$ as

$$\sum_{s=0}^{n-1} \binom{2n-1}{s} \Big[\pi(a)+\pi(b)\Big]^{s} \cdot \Big[1-\pi(a)-\pi(b)\Big]^{2n-1-s},$$

and rewrite the second term of $\mathbb{P}r[\mu(f) \in W_1]$ as

$$\sum_{j=0}^{n-1} \sum_{k=n+1-j}^{2n-1-j} \Bigg[\frac{(2n-1)!\pi(b)^j \pi(a)^k [1-\pi(a)-\pi(b)]^{2n-1-j-k}}{j!k!(2n-1-j-k)!} \times \sum_{t=0}^{n-j-1} \binom{k}{t} G_a^*(x_f)^{k-t} [1-G_a^*(x_f)]^t \Bigg].$$

At the lowest salary in type a's support, $x_f = 0$, the expected payoff is equal to

$$a_2 + (a_1 - a_2) \Bigg[\sum_{s=0}^{n-1} \binom{2n-1}{s} [\pi(a)+\pi(b)]^s [1-\pi(a)-\pi(b)]^{2n-1-s} \Bigg]. \tag{A.39}$$

Since the firm must be indifferent between all of the salaries in the support $[0, \bar{x}_a]$, we can also solve for the value of $\bar{x}_a$ by equating the expected payoffs from $x_f = 0$ and $x_f = \bar{x}_a$. This implies that

$$\bar{x}_a = (a_1 - a_2) \Bigg[\sum_{j=0}^{n-1} \sum_{k=n+1-j}^{2n-1-j} \frac{(2n-1)!\pi(b)^j \pi(a)^k [1-\pi(b)-\pi(a)]^{2n-1-j-k}}{j!k!(2n-1-j-k)!} \Bigg]. \tag{A.40}$$

Moreover, indifference implies that, for all $x_f \in (0, \bar{x}_a)$, we have

$$x_f = (a_1 - a_2) \sum_{j=0}^{n-1} \sum_{k=n+1-j}^{2n-1-j} \Bigg[\frac{(2n-1)!\pi(b)^j \pi(a)^k [1-\pi(a)-\pi(b)]^{2n-1-j-k}}{j!k!(2n-1-j-k)!} \times \sum_{t=0}^{n-j-1} \binom{k}{t} G_a^*(x_f)^{k-t} [1-G_a^*(x)]^t \Bigg]. \tag{A.41}$$

The right-hand side of the equation is a continuous function of $G_a^*(x_f)$, which we have assumed is a continuous function of x_f. Since the left-hand side is strictly increasing in x_f, it follows that there exists a continuous function $G_a^*(x)$ that satisfies the equation.

Now consider type b firms. For any type b firm f, the expected utility of a salary $x_f \in (\bar{x}_a, \bar{x}_b)$ is

$$\mathbb{E}U_b(x_f) = (b_1 - x_f) \cdot \mathbb{P}r[\mu(f) \in W_1]. \tag{A.42}$$

Since type b firms don't care about class W_2 workers, they are either matched with a class W_1 worker, or remain unmatched. In this case, $\mathbb{P}r[\mu(f) \in W_1]$ consists of two different terms,

(1) the probability that there are no more than $n-1$ other type b firms,

(2) the probability that
   - there are more than $n-1$ other type b firms, but
   - the number of type b firms that choose $x > x_f$ is less than or equal to $n-1$.

Again we let j denote the number of type b firms, however now let k denote the number of type b firms that choose $x < x_f$. Then we can write the first term of $\mathbb{P}r[\mu(f) \in W_1]$ as

$$\sum_{j=0}^{n-1} \binom{2n-1}{j} \pi(b)^j [1-\pi(b)]^{2n-1-j}, \tag{A.43}$$

and the second term as

$$\sum_{j=n}^{2n-1} \binom{2n-1}{j} \pi(b)^j [1-\pi(b)]^{2n-1-j} \sum_{k=j-n+1}^{j} \binom{j}{k} G_b^*(x_f)^k [1-G_b^*(x_f)]^{j-k}. \tag{A.44}$$

Furthermore, the second term is 0 for $x_f = \bar{x}_a$ (the lowest salary in type b's support), which means that the equilibrium expected payoff for a type b firm from any $x \in [\bar{x}_a, \bar{x}_b]$ must be

$$(b_1 - \bar{x}_a)\left[\sum_{j=0}^{n-1} \binom{2n-1}{j} \pi(b)^j [1-\pi(b)]^{2n-1-j}\right]. \tag{A.45}$$

At the top of type b's support, the probability that any number of other type b's choose $x < x_f = \bar{x}_b$ is 1. Thus, the expected payoff from choosing $x_f = \bar{x}_b$ is just $b_1 - \bar{x}_b$. In order to ensure indifference, we must have

$$\bar{x}_b = b_1 - (b_1 - \bar{x}_a)\left[\sum_{j=0}^{n-1} \binom{2n-1}{j} \pi(b)^j [1-\pi(b)]^{2n-1-j}\right]. \tag{A.46}$$

Finally, for all $x \in (\bar{x}_a, \bar{x}_b)$, $G_b^*(x)$ must satisfy

$$\frac{b_1 - \bar{x}_b}{b_1 - x} = \Bigg[\sum_{j=0}^{n-1} \binom{2n-1}{j} \pi(b)^j [1 - \pi(b)]^{2n-1-j} \tag{A.47}$$

$$+ \quad \sum_{j=n}^{2n-1} \binom{2n-1}{j} \pi(b)^j [1 - \pi(b)]^{2n-1-j} \sum_{k=j-n+1}^{j} \binom{j}{k} G_b^*(x_f)^k [1 - G_b^*(x_f)]^{j-k} \Bigg].$$

□