Pricing American Options Under GARCH Model

Project Report for MAP 5615-0001
Monte Carlo Methods in Financial Mathematics

Hua-Yi Lin *

April 29, 2015

Abstract

This project gives a review of the least-squares Monte Carlo (LSM) American-styled option valuation method as well as the generalized autoregressive conditional heteroskedastic (GARCH) option pricing model. There are two main kinds of LSM pricing methods: one is proposed by Longstaff and Schwartz (2001), and the other is proposed by Tsitsiklis and Van Roy (2001). A summary and comparison of these two methods is presented in this work. Numerical results show that Longstaff and Schwartz's approach not only produces results comparable to those calculated by the finite difference approach but also converges very quickly, while Tsitsiklis and Van Roy's method often generates highly-biased results, especially for out-of-the-money contracts. Finally, I apply the LSM method to price American options under the GARCH model, and get a comparable result to those presented in early studies.

Keywords American Options; GARCH Process; Least-Squares Monte Carlo; Option Pricing

*E-mail: `hlin@math.fsu.edu`.

1 Introduction

Simulation-based methods have become one of the most popular approaches to evaluate financial derivatives due to their good flexibility. Simulation allows state variables to follow more general stochastic processes than the geometric Brownian motion, such as jump-diffusion models or stochastic volatility models. Also, it is readily applied to evaluate derivatives which possess both path-dependent and American-style features. It also works for pricing derivatives depending on multiple risk factors. In sum, simulation is suitable for dealing with complicated derivative pricing problems.

The Monte Carlo method is one of the most widely used simulation techniques among all simulation-based methods. It is readily used to price European-style derivatives, especially when the underlying asset is assumed to follow a complicated dynamic. However, American-style products pose a significant challenge to the Monte Carlo method because of the difficulty in estimating the early exercise boundary, or equivalently the optimal exercising time. Several researchers have devoted their efforts to dealing with this problem. For example, the random-tree method proposed by Broadie and Glasserman (1997), the quantization method, or say the state-space partitioning method, proposed by Bally et al. (2003), and the least-squares-based method proposed by Tsitsiklis and Van Roy (2001) and Longstaff and Schwartz (2001).

The least-squares Monte Carlo (LSM hereafter) method is a simple yet powerful tool for pricing American-style securities. It approximates an option's continuation value via ordinary least-squares. Moreover, it allows practitioners to choose suitable basis functions used in regressions based on their knowledge or intuition about the problem. The accuracy of LSM often depends on the basis functions used in regression.

The major differences between Tsitsiklis and Van Roy (2001) and Longstaff and Schwartz (2001) is that Longstaff and Schwartz (2001) use only in-the-money paths to estimate the least-squares coefficients, while Tsitsiklis and Van Roy (2001) use all the sample paths. Moreover, Tsitsiklis and Van Roy (2001) use the maximum of the continuation value estimated by regression and the early-exercise payoff as the estimated option price. Longstaff and Schwartz (2001) estimate the value of each currently in-the-money options by its early-exercise payoff when the early-exercise value is greater than the estimated continuation value; as to currently at-the-money or out-of-the-money options, the option price is set to the discounted value of their previous estimated values. The advantage of Longstaff and Schwartz's method is that it won't take "zero" as an estimation of option value unless the option is out-of-the-money at maturity. Accordingly Longstaff and Schwartz (2001) is computationally more efficient and often generates better results, especially for out-of-the-money contracts.

Finally, I demonstrate that how the LSM method can be applied to price American options with underlying asset process follows The generalized autoregressive conditional heteroskedastic (GARCH hereafter) model . The GARCH model studied by Bollerslev (1986) has gained much popularity for option pricing since it can explain some well-documented systematic biases associated with the Black-Scholes model. For example, underpricing of out-of-the-money options, underpricing of options on low-volatility underlying assets, and the volatility smile phenomenon. Moreover, if we set the variance to be homoskedastic in the

GARCH pricing model, it reduces to the Black-Scholes model.

It is worth noting that due to the complexity of the GARCH model, it is not easy to transform the GARCH process to the risk-neutral world. Fortunately, we can still have a local version of risk-neutralization by exploiting the locally risk-neutral valuation principle (LRNVP hereafter) studied by Duan (1995). . This project also present a review of the GARCH process and the LRNVP.

2 Valuation of American Options through the Dynamic Programming Approach

Consider an American option with an underlying asset S which follows a Markov process. Let h_m denote the payoff function for exercise at time t_m and $V_m(x)$ denote the option price at time t_m conditional on the option has not been exercised before time t_m. The option price can be determined backwardly via the following dynamic programming recursions:

$$V_M(x) = h_M(x) \tag{1}$$

$$V_{m-1}(x) = \max\{h_{m-1}(x),\ \mathbb{E}[D_{m-1}(S_i)V_m(S_i)|S_{i-1} = x]\}, \tag{2}$$

where $m = M, M-1, \cdots, 1$; $D_{i-1}(\cdot)$ is the discount factor from t_m to t_{m-1}. The conditional expectation is taken under the risk-neutral measure. The continuation value of an American option is the value of holding rather than exercising the option.

The continuation value of the American option at time t_m is the value of holding the option from t_m to t_{m+1} rather than exercising it at time t_m. Also, it is the second term in the maximum function:

$$CV_m(x) = \mathbb{E}[D_{m-1}(S_i)V_m(S_m)|S_{i-1} = x].$$

The value of an American option is the value achieved by exercising optimally. Finding this value entails finding the optimal exercise rule, see e.g. Glasserman (2003). If we have a method to estimate the continuation value of an option, we can accordingly determine an optimal-stopping rule via the comparison of the early-exercise value and the estimated continuation value. In other words, a stopping rule $\widehat{\tau}$ can be determined by

$$\widehat{\tau} = \min\left\{m \in \left\{1 \cdots M\right\} : h_m(S_i) \geq \widehat{CV}_m(S_i)\right\},$$

where $\widehat{CV}_m(S_i)$ is the estimation of continuation value at time t_m.

In sum, in order to evaluate American-style securities, we need a systematic and effective way to estimate its continuation value. The LSM method studied by Longstaff and Schwartz (2001) and Tsitsiklis and Van Roy (2001) exploits the ordinary least-squares regression to give an estimation method which meets these two requirements.

3 A Simple Example: Using LSM to Price American Options

In this section, I provide a simple example to illustrate the idea of pricing American options based on the least-squares Monte Carlo simulation approach. Con-

sider an American put option on a share of non-dividend paying stock S which follows a geometric Brownian motion. The initial stock price is $S_0 = 36$, strike price is $K = 40$, volatility is $\sigma = 0.2$, risk-free interest rate is $r = 0.06$, and time to maturity is $T = 1$. Moreover, we divide T into 2 equally-spaced time periods and let t_0, t_1, and t_2 be the boundary point of them. Note that in this setting t_0 is the current time and t_2 is the maturity date of the put option. We assume the American put option is only exercisable at t_1 and t_2. For simplicity, six paths (3+ 3 antithetic) are generated in order to illustrate the idea of the LSM approach:

path	S_0	S_1	S_2
1	36	38.2438	49.2277
2	36	48.8551	45.9914
3	36	36.8913	36.7263
4	36	36.5639	38.2271
5	36	27.6101	30.5261
6	36	35.2708	28.5193

Conditional on not exercising the option before the maturity date, the cash flow realized by the option holder at time 2, V_2, will be $\max(40 - S_2, 0)$:

path	S_2	V_2
1	49.2277	0.0000
2	45.9914	0.0000
3	36.7263	3.2737
4	38.2271	1.7729
5	30.5261	9.4739
6	28.5193	11.4807

Now we move backwards to time t_1. At time t_1, if the put option is in-the-money, an option holder must decide whether to exercise the option or not based on the scale of its continuation value and early exercise value. If the early exercise value is greater than the continuation value, then the option holder will choose to exercise the option immediately; otherwise the option will be held to the next time point.

3.1 Longstaff and Schwartz (2001)

Longstaff and Schwartz (2001) suggest using only those paths which are in-the-money at time t_1 to estimate the continuation value of an option via the ordinary least-squares. At time t_1, the put option is in-the-money for path 1, 3, 4, 5 and 6:

path	S_1	$e^{-r*\Delta t} \cdot V_2$	in the money
1	38.2438	0.0000	v
2	48.8551	0.0000	
3	36.8913	3.1770	v
4	36.5639	1.7205	v
5	27.6101	9.1939	v
6	35.2708	11.1414	v

Where $\Delta t = 2/T = t_2 - t_1 = t_1 - t_0$ is the time duration between two exercisable points. We regress the discounted cashflows $e^{-r*\Delta t} \cdot V_2$ on a constant, S_1, and S_1^2 for those in-the-money paths, and get the estimated continuation value fucntion:

$$\widehat{CV}(S_1) = -293.475183 + 19.614722S_1 - 0.313245S_1^2.$$

Now we can get the estimated continuation value by plugging S_1 into the regression line $\widehat{CV}(\cdot)$:

path	S_1	$\widehat{CV}(S_1)$	$K - S_1$	early exercise
1	38.2438	-1.4824	1.7562	o
2				
3	36.8913	3.8206	3.1087	x
4	36.5639	4.9322	3.4361	x
5	27.6101	9.2968	12.3899	o
6	35.2708	8.6655	4.7292	x

For path 1 and 5, it is optimal to exercise the put option at t_1 since its early exercise value is greater than the estimated continuation value.

The cashflows at time t_1 will be the greater of early exercise value and the discounted cashflow at time t_2:

path	$e^{-r*\Delta t} \cdot V_2$	early exercise value	V_1
1	0.0000	1.7562	1.7562
2	0.0000	-	0.0000
3	3.1770	-	3.1770
4	1.7205	-	1.7205
5	9.1939	12.3899	12.3899
6	11.1414	-	11.1414

Finally, the put price will be the average of the discounted values V_1:

$$\begin{aligned} \text{Put Value} &= e^{-.06*.5}\left(\frac{1.7562 + 0 + 3.1770 + 1.7205 + 12.3899 + 11.1414}{6}\right) \\ &= 4.882142. \end{aligned}$$

3.2 Tsitsiklis and Van Roy (2001)

On the other hand, Tsitsiklis and Van Roy (2001) suggests using the information in "all sample paths" to estimate the continuation value by ordinary least-squares:

path	S_1	$e^{-r*\Delta t} \cdot V_2$
1	38.2438	0.0000
2	48.8551	0.0000
3	36.8913	3.1770
4	36.5639	1.7205
5	27.6101	9.1939
6	35.2708	11.1414

Thus we get a different regression line for the continuation value:

$$\widehat{CV}(S_1) = 44.169749 - 1.651056S_1 + 0.015095S_2^2.$$

Then we can set V_2 to the greater of the estimated continuation value and the early exercise value:

path	$\widehat{CV}(S_1)$	$\max(K - S_1, 0)$	V_1
1	3.1050	1.7562	3.1050
2	-0.4635	0.0000	0.0000
3	3.8040	3.1087	3.8040
4	3.9816	3.4361	3.9816
5	10.0912	12.3899	12.3899
6	4.7144	4.7292	4.7292

Finally, the put price will be the average of the discounted values V_1:

$$\begin{aligned} \text{Put Value} &= e^{-.06*.5}\left(\frac{3.1050 + 0 + 3.8040 + 3.9816 + 12.3899 + 4.7292}{6}\right) \\ &= 4.530317. \end{aligned}$$

4 The GARCH Option Pricing Model

Consider a discrete-time economy and let S_t be the asset price at time $t \in \{0, 1, \cdots, T\}$. Its conditional one-period rate of return is assumed to be lognormally distributed under probability measure P. More precisely,

$$\ln \frac{S_t}{S_{t-1}} = r + \lambda\sigma_t - \frac{1}{2}\sigma_t^2 + \sigma_t\epsilon_t,$$

where ϵ_t has mean zero and conditional variance 1 under measure P; r is the constant one-period risk-free return; λ is the constant unit risk premium; h_t follows the GARCH(p,q) process. Namely,

$$\epsilon_t|\mathcal{F}_{t-1} \sim N(0, 1),$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i(\sigma_{t-i}\epsilon_{t-i})^2 + \sum_{i=1}^{p} \beta_i\sigma_{t-i}^2.$$

In order to develop the GARCH option pricing model, the conventional risk-neutral valuation relationship has to be generalized to accommodate heteroskedasticity of the asset return process. Thus, we need a generalized version of this principle.

A pricing measure Q is said to satisfy the locally risk-neutral valuation principle (LRNVP) if

- The pricing measure Q is mutually absolutely continuous with respect to measure $\mathbb{P}$.
- $S_t/S_{t-1}|\mathcal{F}_{t-1}$ is lognormally distributed under Q.
- $\mathbb{E}^Q(S_t/S_{t-1}|\mathcal{F}_{t-1}) = e^r$.
- $\mathrm{Var}^Q(\ln(S_t/S_{t-1})|\mathcal{F}_{t-1}) = \mathrm{Var}^P(\ln(S_t/S_{t-1})|\mathcal{F}_{t-1})$.

When the one-period return and the stochastic discount factor are both (conditionally) lognormally distributed, then the pricing measure Q satisfes the LRNVP. Let d_{t+1} be the one-period discount factor from time $t+1$ to t. Then, the standard pricing result gives

$$S_t = E^P\left[S_{t+1}d_{t+1}|\mathcal{F}_t\right].$$

Define a new measure Q by $dQ = e^{rT}\prod_{i=1}^{T} d_i dP$. Then Q is a probability measure, since

$$\begin{aligned}
\int 1\, dQ &= \int e^{rT}\prod_{i=1}^{T} d_i dP = \mathbb{E}^P\left[e^{rT}\prod_{i=1}^{T} d_i \middle| \mathcal{F}_0\right] \\
&= \mathbb{E}^P\left[e^{rT}\prod_{i=1}^{T-1} d_i\, \mathbb{E}^P\left[d_T \middle| \mathcal{F}_{T-1}\right] \middle| \mathcal{F}_0\right] \\
&= \mathbb{E}^P\left[e^{r(T-1)}\prod_{i=1}^{T-1} d_i \middle| \mathcal{F}_0\right] \\
&= \cdots = \mathbb{E}^P\left[1 \middle| \mathcal{F}_0\right] = 1.
\end{aligned}$$

Moreover, the conditional expected return under Q is the risk-free rate.

$$\begin{aligned}
\mathbb{E}^Q\left[S_t|\mathcal{F}_{t-1}\right] &= \frac{\mathbb{E}^P\left[S_t\mathbb{E}^P\left[e^{rT}\prod_{i=1}^{T} d_i \middle| \mathcal{F}_t\right] \middle| \mathcal{F}_{t-1}\right]}{\mathbb{E}^P\left[e^{rT}\prod_{i=1}^{T} d_i \middle| \mathcal{F}_{t-1}\right]} \\
&= \frac{\mathbb{E}^P\left[S_t e^{r(T-t)} \middle| \mathcal{F}_{t-1}\right]}{e^{r(T-t-1)}} \\
&= e^r\mathbb{E}^P[S_t d_t|\mathcal{F}_{t-1}] \\
&= e^r S_{t-1}
\end{aligned}$$

$$\therefore \mathbb{E}^Q\left[\frac{S_t}{S_{t-1}}\right] = e^r.$$

Assume further that the discount factor is (conditional) lognormally distributed:

$$\ln(d_t) = a_t + b_t\epsilon_t + z_t,$$

where z_t is independent of ϵ_t and standard normally distributed. We can compute the conditional moment generating function of the return innovation under measure Q as follows:

$$\begin{aligned}
&\mathbb{E}^Q\left[e^{c\epsilon_t} \middle| \mathcal{F}_{t-1}\right] = \mathbb{E}^P\left[e^{c\epsilon_t}e^r d_t \middle| \mathcal{F}_{t-1}\right] = \mathbb{E}^P\left[e^{r+a_t+z_t+(b_t+c)\epsilon_t} \middle| \mathcal{F}_{t-1}\right] \\
&= q_t \exp\left(\frac{c^2}{2} + b_t c\right),
\end{aligned}$$

where q_t doesn't contain c. Setting $c = 0$ gives rise to $q_t = 1$, and therefore

$$\mathbb{E}^Q\left[e^{c\epsilon_t} \middle| \mathcal{F}_{t-1}\right] = \exp\left(\frac{c^2}{2} + b_t c\right).$$

Thus, ϵ_t is a Q conditional normal random variable with mean b_t and variance 1. b_t can then be determined by applying the risk-neutrality condition.

Theorem 1. *The LRNVP implies that, under pricing measure Q,*

$$\ln \frac{S_t}{S_{t-1}} = r - \frac{1}{2}\sigma_t^2 + \sigma_t \xi_t,$$

where

$$\xi_t | \mathcal{F}_{t-1} \sim N(0,1)$$

and

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i ((\xi_{t-i} - \lambda)\sigma_{t-i})^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2.$$

Proof.

$$\because \frac{X_t}{X_{t-1}}\bigg| \mathcal{F}_t \sim \text{lognormal} \quad \therefore \ln \frac{X_t}{X_{t-1}} = \nu_t + \xi_t \sigma_t,$$

where ν_t is the conditional mean; $\xi_t \sim N(0,1)$ under Q.

$$\mathbb{E}^Q \left[\frac{X_t}{X_{t-1}} \bigg| \mathcal{F}_{t-1} \right] = \mathbb{E}^Q \left[e^{\nu_t + \xi_t \sigma_t} \bigg| \mathcal{F}_{t-1} \right] = e^{\nu_t + \sigma_t^2/2} = e^r,$$

where $\sigma_t^2 = \text{Var}^Q(\ln(S_t/S_{t-1})|\mathcal{F}_{t-1}) = \text{Var}^P(\ln(S_t/S_{t-1})|\mathcal{F}_{t-1})$ by LRNVP. Therefore $\nu_t = r - \sigma^2/2$. Moreover,

$$r + \lambda\sigma_t - \frac{1}{2}\sigma_t^2 + \epsilon_t \sigma_t = r - \frac{1}{2}\sigma_t^2 + \xi_t \sigma_t \quad \therefore \epsilon_t = \xi_t - \lambda. \quad \square$$

5 Description of Algorithms

5.1 Generating GARCH(1,1) Stock Price Paths

Algorithm 1 Generating GARCH(1,1) stock price paths.

```
Given α_0, α_1, β_0, λ, and σ_0.
for i = 1 ··· N do
    ξ_0 = σ_0 * N(0,1)
    σ_1 = sqrt(α_0 + α_1 (ξ_0 − λσ_0)^2 + β_1 σ_0^2)
    ξ_1 = σ_1 * N(0,1)
    S_{i,1} = S_0 exp(r * Δt − σ_1^2/2 + ξ_1)
    σ_0 = σ_1
    ξ_0 = ξ_1
    for j = 2 ··· M do
        σ_1 = sqrt(α_0 + α_1 (ξ_0 − λσ_0)^2 + β_1 σ_0^2)
        ξ_1 = σ_1 * N(0,1)
        S_{i,j} = S_{i,j−1} exp(r * Δt − σ_1^2/2 + ξ_1)
        σ_0 = σ_1
        ξ_0 = ξ_1
    end for
end for
```

5.2 Least-Squares Monte Carlo

Algorithm 2 LSM algorithm by Tsitsiklis and Van Roy (2001).

1. Generate N stock paths with M time steps, where M is the number of exercisable times: $\left\{S_m^{(n)} : n = 1, \cdots, N; m = 1, \cdots, M\right\}$.

2. For each path, set the value function to the option's payoff at the end nodes: $\left\{V(S_M^{(n)}) = h_M(S_M^{(n)}),\ n = 1, \cdots, N\right\}$.

3. **FOR** $m = M - 1$ **to** 1
   • Discount function by one time step:

   $$V(S_m^{(n)}) = e^{-r\Delta t} V(S_{m+1}^{(n)}),\ n = 1, \cdots, N.$$

   • Solve the linear least-squares problem $\min_x ||b - Ax||_2$, where

   $$b = \begin{pmatrix} V(S_m^{(1)}) \\ V(S_m^{(2)}) \\ \vdots \\ V(S_m^{(N)}) \end{pmatrix}, \quad A = \begin{pmatrix} 1 & S_m^{(1)} & (S_m^{(1)})^2 & \cdots & (S_m^{(1)})^{nb-1} \\ 1 & S_m^{(2)} & (S_m^{(2)})^2 & \cdots & (S_m^{(2)})^{nb-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & S_m^{(N)} & (S_m^{(N)})^2 & \cdots & (S_m^{(N)})^{nb-1} \end{pmatrix}.$$

   Note that $x^* = (A^T A)^{-1} A^T b$.
   • Calculate the estimated continuation value for all paths:

   $$\widehat{CV}(S_m^{(n)}) = x_0 + x_1 S_m^{(n)} + \cdots + x_{nb-1}(S_m^{(n)})^{nb-1},\ n = 1, \cdots, N.$$

   • Set $V(S_m^{(n)}) = \max(h_m(S_m^{(n)}),\ \widehat{CV}(S_m^{(n)})), \quad n = 1, \cdots, N$
   END FOR

4. Option price$= e^{-r\Delta t}(V(S_1^{(1)}) + V(S_1^{(2)}) + \cdots + V(S_1^{(N)})/N$.

Algorithm 3 LSM algorithm by Longstaff and Schwartz (2001).

1. Generate N stock paths with M time steps, where M is the number of exercisable times: $\left\{S_m^{(n)} : n = 1, \cdots, N;\ m = 1, \cdots, M\right\}$.

2. For each path, set the value function to the option's payoff at the end nodes: $\left\{V(S_M^{(n)}) = h_M(S_M^{(n)}),\ n = 1, \cdots, N\right\}$.

3. **FOR** $m = M - 1$ **to** 1

   • Discount the value function for one time step:

   $$V(S_m^{(n)}) = e^{-r\Delta t} V(S_{m+1}^{(n)}),\ n = 1, \cdots, N$$

   • Find all paths that are in-the-money at time m:

   $$\left\{S_m^{(n_k)} : k = 1, \cdots, j\right\}$$

   • Solve the linear least-squares problem $\min_x ||b - Ax||_2$, where

   $$b = \begin{pmatrix} V(S_m^{(n_1)}) \\ V(S_m^{(n_2)}) \\ \vdots \\ V(S_m^{(n_j)}) \end{pmatrix}, \quad A = \begin{pmatrix} 1 & S_m^{(n_1)} & (S_m^{(n_1)})^2 & \cdots & (S_m^{(n_1)})^{nb-1} \\ 1 & S_m^{(n_2)} & (S_m^{(n_2)})^2 & \cdots & (S_m^{(n_2)})^{nb-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & S_m^{(n_j)} & (S_m^{(n_j)})^2 & \cdots & (S_m^{(n_j)})^{nb-1} \end{pmatrix}.$$

   Note that the solution for this least-squares problem can be expressed as $x^* = (A^T A)^{-1} A^T b$.

   • Calculate the estimated continuation value for in-the-money paths:

   $$\widehat{CV}(S_m^{(n_k)}) = 1 + S_m^{(n_k)} + (S_m^{(n_k)})^2 + \cdots + (S_m^{(n_k)})^{nb-1},\ k = 1, \cdots, j.$$

   • Set

   $$V(S_m^{(n_k)}) = h_m(S_m^{(n_k)}) \quad \text{if} \quad h_m(S_m^{(n_k)}) \geq \widehat{CV}(S_m^{(n_k)}) \geq 0,\ k = 1, \cdots, j.$$

   END FOR

4. Option price$= e^{-r\Delta t}(V(S_1^{(1)}) + V(S_1^{(2)}) + \cdots + V(S_1^{(N)})/N$.

6 Numerical Results

Table 1: American Put Option Prices Under Geometric Brownian Motion.

S_0	σ	T (year)	Finite Diff.	Longstaff Schwartz	Time (sec.)	Diff.	Tsitsiklis Van Roy	Time (sec.)	Diff.
36	0.2	1	4.478	4.478	15.977	0.000	4.771	36.34	0.293
36	0.2	2	4.840	4.841	28.99	0.001	5.281	73.84	0.441
36	0.4	1	7.101	7.100	14.757	-0.001	8.043	37.56	0.942
36	0.4	2	8.508	8.503	28.509	-0.005	10.734	75.86	2.226
38	0.2	1	3.250	3.250	13.502	0.000	3.581	36.26	0.331
38	0.2	2	3.745	3.739	24.754	-0.006	4.340	73.74	0.595
38	0.4	1	6.148	6.145	13.246	-0.003	7.151	36.23	1.003
38	0.4	2	7.670	7.661	26.003	-0.009	9.849	73.62	2.179
40	0.2	1	2.314	2.312	10.262	-0.002	2.694	36.17	0.380
40	0.2	2	2.885	2.883	19.888	-0.002	3.687	73.58	0.802
40	0.4	1	5.312	5.313	11.537	0.001	6.790	36.25	1.478
40	0.4	2	6.920	6.921	23.491	0.001	9.830	73.91	2.910
42	0.2	1	1.617	1.611	7.288	-0.006	2.091	36.29	0.474
42	0.2	2	2.212	2.210	15.444	-0.002	3.051	73.60	0.839
42	0.4	1	4.582	4.573	9.867	-0.009	5.791	36.43	1.209
42	0.4	2	6.248	6.247	21.092	-0.001	9.544	73.73	3.296
44	0.2	1	1.110	1.106	5.358	-0.004	1.660	36.09	0.550
44	0.2	2	1.690	1.687	12.306	-0.003	2.642	72.66	0.952
44	0.4	1	3.948	3.946	8.459	-0.002	5.533	36.37	1.585
44	0.4	2	5.647	5.632	18.868	-0.015	9.048	73.72	3.401

The "Finite Diff." column shows the American put prices calculated via the finite difference method, which comes from Table 1 of Longstaff and Schwartz (2001) and can play the role of benchmark prices. The columns named by "Longstaff Scharez" and "Tsitsiklis Van Roy" show the American put prices simulated by the LSM algorithms described in Longstaff and Schwartz (2001) and Tsitsiklis and Van Roy (2001), respectively. The simulation is based on 100,000 (50,000+50,000 antithetic) paths. Four monomial basis function (i.e. 1, S, S^2, and S^3) are used. In this comparisons, the strike price K=40, risk-free interest rate $r = 0.06$. Initial stock price S_0, volatility σ, and time to maturity T are as indicated. The column "Time(sec.)" shows the computational time.

Table 2: At-The-Monty American Put Option Prices Under GARCH(1,1) Process.

T	Basis 1			Basis 2		
(days)	Stentoft	LSM	diff.	Stentoft	LSM	diff.
2	0.5641	0.5589	0.0052	0.5638	0.5589	0.0049
10	1.1905	1.1930	-0.0025	1.1914	1.1930	-0.0016
50	2.3954	2.3984	-0.0030	2.4026	2.3984	0.0042
100	3.1377	3.1443	-0.0066	3.1386	3.1443	-0.0057

Comparison of least-squares Monte Carlo put prices in Stentoft (2005), with $S_0 = K = 100$, $r = 0.1$. The left part and right part use different basis in LSM. Basis 1 is $\{1, S_t, S_t^2, \sqrt{h_t}, h_t, S_t\sqrt{h_t}\}$. Basis 2 is $\{1, S_t, S_t^2, S_t^3\}$. The GARCH(1,1) parameters are: $\alpha_0 = 6.575 \times 10^{-6}$, $\alpha_1 = .04$, $\beta_1 = .9$, and $\lambda = 0$.

7 Conclusions

This project clearly explains the steps for the Least-Squares Monte Carlo (LSM) American option pricing method. A comparison of two different LSM methods is made in this work. Numerical results show that the method proposed by Longstaff and Schwartz (2001) is not only more accurate but also more efficient than the one proposed by Tsitsiklis and Van Roy (2001). Finally, the LSM algorithm is applied to price American options under the GARCH model. Numerical results show that Longstaff and Schwartz (2001)'s algorithm still generates reasonable pricing results even if the underlying asset is assumed to follow the GARCH(1,1) process.

References

Bally, V., G. Pages, et al. (2003). A quantization algorithm for solving multi-dimensional discrete-time optimal stopping problems. *Bernoulli 9*(6), 1003–1049.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of econometrics 31*(3), 307–327.

Broadie, M. and P. Glasserman (1997). Pricing American-style securities using simulation. *Journal of Economic Dynamics and Control 21*(8), 1323–1352.

Duan, J.-C. (1995). The GARCH option pricing model. *Mathematical Finance 5*(1), 13–32.

Glasserman, P. (2003). *Monte Carlo methods in financial engineering*, Volume 53. Springer Science & Business Media.

Longstaff, F. A. and E. S. Schwartz (2001). Valuing American options by simulation: A simple least-squares approach. *Review of Financial Studies 14*(1), 113–147.

Stentoft, L. (2005). Pricing American options when the underlying asset follows garch processes. *Journal of Empirical Finance 12*(4), 576–611.

Tsitsiklis, J. N. and B. Van Roy (2001). Regression methods for pricing complex american-style options. *Neural Networks, IEEE Transactions on 12*(4), 694–703.