Solving RTM using Discrete Ordinate Method (DOM)
and
How it is implemented in MWRT

The radiative transfer model

$$\mu \frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{\omega_0}{2}\int_{-1}^{+1} I(\tau,\mu')P(\mu,\mu')d\mu' - (1-\omega_0)B(\tau) \qquad (1)$$

$I(\tau,\mu)$ is radiance, may be convert to brightness temperature using Planck's function. μ is the cosine of zenith angle. τ is optical depth (τ=0 at TOA, and increases downward). ω_0 is single-scattering albedo. P is phase function; B is Planck's function. The calculation of $\tau=\sigma_{ext}\Delta z$ and $\omega_0=\sigma_{sca}/\sigma_{ext}$ may follow standard Mie theory. (absub.f, kcoef.f)
In MWRT, P is assumed to follow Henyey-Greenstein equation, and is expended with Legendre polynomial (p_l):

$$P(\cos\Theta) = \frac{1-g^2}{(1+g^2-2g\cos\Theta)^{\frac{3}{2}}} = \sum_{\ell=0}^{N}(2\ell+1)g^{\ell}p_{\ell}(\cos\Theta) \equiv \sum_{\ell=0}^{N} A_{\ell}p_{\ell}(\cos\Theta)$$

Asymmetry factor g is calculated following Mie theory (Book: Bohren and Huffman, 1983, p120). (kcoef.f, hgpf.f)
For azimuth-independent case (spheres, randomly-orientated irregular particles), the cosine of the scattering angle cosΘ can be denoted as $\mu\mu'$ (Liou, 1974, JAS, Vol 31, p1473), so that

$$P(\cos\Theta) = P(\mu,\mu') = \sum_{\ell=0}^{N} A_{\ell}p_{\ell}(\mu)p_{\ell}(\mu')$$

N is the number of terms to add. In DOM, N=2n-1 (2n is the stream number, see text below). To minimize the error associated with this cutoff, a δ-adjustment (scaling g, τ, and ω_0) is applied (radsub.f). The δ-adjustment is well explained in Liou (1992, Book).

Additionally, $B(\tau)$ in MWRT is assumed to be $B_0+B_1\tau$, i.e., linearly varying from the top of the layer (B_0) to the bottom of the layer.

To solve the equation using DOM, first write the equation in discrete form (*i*=-n, n):

$$\mu_i \frac{dI(\tau,\mu_i)}{d\tau} = I(\tau,\mu_i) - \frac{\omega_0}{2}\sum_{j=-n}^{n} a_j I(\tau,\mu_j)P(\mu_i,\mu_j) - (1-\omega_0)(B_0+B_1\tau) \qquad (2)$$

where μ_i and μ_j are the Gaussian points and a_j is the weights at μ_j. 2n is the "stream" number. For 4-stream model, n=2.

Let $C_{ij} = \frac{\omega_0}{2} a_j P(\mu_i, \mu_j) = \frac{\omega_0}{2} a_j \sum_{\ell=0}^{N} A_\ell p_\ell(\mu_i) p_\ell(\mu_j)$,

and

$$b_{ij} = \begin{array}{ll} C_{ij} / \mu_i & (i \neq j) \\ (C_{ij} - 1) / \mu_i & (i = j) \end{array}$$

Then, (2) becomes

$$\mu_i \frac{dI(\tau, \mu_i)}{d\tau} = -\sum_{j=-n}^{n} b_{ij} I(\tau, \mu_i) - (1 - \omega_0)(B_0 + B_1 \tau) \qquad (3)$$

Because i=-n, n, we have 2n such equations. The problem becomes solving the 2n equations given boundary conditions. The solution should be "a linear combination of all solutions of the homogenous equation (without the last term)" plus one "special solution".

The solutions of the homogenous equations takes the form: $W(\mu_i)\exp(-k\tau)$, where k and W are the eigenvalues and eigenvectors. It has been proved that k is real and comes in pairs (+ and -). For $n\leq 2$, k and W may be evaluated analytically (strmeig.f, also refer Liou 1992, Book). For higher streams, they must be evaluated numerically (refer Lapack routines in ~/com directory). A special solution for (3) can be written as $q(\mu_i)+B_1\tau$, where q can be evaluated by solving (pqsolver.f)

$$\sum_{j=-n}^{n} b_{ij} q(\mu_i) = -(1 - \omega_0) B_0 / \mu_i - B_1$$

Now, we may write the solution of (3):

$$I(\tau, \mu_i) = \sum_{j=-n}^{n} L_j W_j(\mu_i) \exp(-k_j \tau) + q(\mu_i) + B_1 \tau \qquad (4)$$

L_j can be determined using boundary condition and the continuity of radiances between layers. However, as mentioned earlier, k is in pairs with both positive and negative values. If τ is large, the term $\exp(-k\tau)$ and $\exp(k\tau)$ will be different by several orders, which poses a difficult challenge to solve the equation mathematically. To avoid this problem, we further let (j=1,2 for 4-stream)

$L_j=(M_j+N_j)/2$
$L_{-j}=(M_j-N_j)\exp(-k_j\tau^*)/2$

where τ^* is the optical depth of an entire layer (Note τ is the optical depth at any location within the layer). (4) is then written as (i=-2,-1,1,2 for 4-stream)

$$I(\tau,\mu_i)=\sum_{j=1}^{n}\left[M_j Y_{ij}(\tau)+N_j Z_{ij}(\tau)\right]+q(\mu_i)+B_1\tau\,, \tag{5}$$

where

$$Y_{ij}(\tau)=\tfrac{1}{2}\left[W_j(\mu_i)\exp(-k_j\tau)+W_{-j}(\mu_i)\exp[-k_j(\tau^*-\tau)]\right]$$
$$Z_{ij}(\tau)=\tfrac{1}{2}\left[W_j(\mu_i)\exp(-k_j\tau)-W_{-j}(\mu_i)\exp[-k_j(\tau^*-\tau)]\right]$$

and k_js are positive eigenvalues. This way, no large exp terms will be in (5). The next task is to build the equations using boundary condition and continuity between layers to solve M_j and N_j.

At the top of the 1 st layer (counting layer from top down, i.e., first layer is at TOA): $I(0,-\mu_i)=I_0$. I_0 is the Plank's function corresponding to 3 K. That is (*i*=1, 2 for 4-stream),

$$\sum_{j=1}^{n}\left[M_j^{(1)}Y_{-ij}^{(1)}(0)+N_j^{(1)}Z_{-ij}^{(1)}(0)\right]=I_0-q^{(1)}(\mu_{-i}) \tag{6}$$

Continuity between layer *l* and *l*+1:

(upward)

$$\sum_{j=1}^{n}\left[M_j^{(\ell)}Y_{ij}^{(\ell)}(\tau^*)+N_j^{(\ell)}Z_{ij}^{(\ell)}(\tau^*)-M_j^{(\ell+1)}Y_{ij}^{(\ell+1)}(0)-N_j^{(\ell+1)}Z_{ij}^{(\ell+1)}(0)\right]=q^{(\ell+1)}(\mu_i)-q^{(\ell)}(\mu_i)-B_1^{(\ell)}(\tau^*) \tag{7}$$

(downward)

$$\sum_{j=1}^{n}\left[M_j^{(\ell)}Y_{-ij}^{(\ell)}(\tau^*)+N_j^{(\ell)}Z_{-ij}^{(\ell)}(\tau^*)-M_j^{(\ell+1)}Y_{-ij}^{(\ell+1)}(0)-N_j^{(\ell+1)}Z_{-ij}^{(\ell+1)}(0)\right]=q^{(\ell+1)}(\mu_{-i})-q^{(\ell)}(\mu_{-i})-B_1^{(\ell)}(\tau^*) \tag{8}$$

Lower boundary (surface) with Fresnel surface ($I^+=\varepsilon I_s+RI^-$):

$$\sum_{j=1}^{n}\left\{M_j^{(L)}\left[Y_{ij}^{(L)}(\tau^*)-R_iY_{-ij}^{(L)}(\tau^*)\right]+N_j^{(L)}\left[Z_{ij}^{(L)}(\tau^*)-R_iZ_{-ij}^{(L)}(\tau^*)\right]\right\}=(1-R_i)[I_s-q^{(L)}(\mu_{-i})-B_1^{(L)}\tau^*] \tag{9-1}$$

where I_s is the Planck's function with surface temperature T_s. R_i is the reflectivity (1-ε) at direction μ_i.

Lower boundary with Lambertian surface ($I^+=\varepsilon I_s+2R\int_0^1 I(-\mu)\mu d\mu$):

$$\sum_{j=1}^{n}\left\{M_j^{(L)}\left[Y_{ij}^{(L)}(\tau^*)-2R\sum_{k=1}^{n}Y_{-ik}^{(L)}(\tau^*)\mu_k a_k\right]+N_j^{(L)}\left[Z_{ij}^{(L)}(\tau^*)-2R\sum_{k=1}^{n}Z_{-ik}^{(L)}(\tau^*)\mu_k a_k\right]\right\}$$

$$=(1-R)[I_s-B_1^{(L)}\tau^*]-q^{(L)}(\mu_i)+2R\sum_{k=1}^{n}q^{(L)}(\mu_{-k})\mu_k a_k$$

(9-2)

(6)-(9) form a 2nL by 2nL matrix (L is number of layers) with 2nL unknowns (M and N) on the left and 2nL known values on the right. The equations can be solved by standard mathematical routines (Lapack). It is also noted that the left-hand matrix is a banded diagonal matrix, so memory and computation time may be saved if we use a routine specially designed for this type of matrix. Arranging and solving the equations are done in gettb.f.

After solving (6)-(9), M and N are obtained. If we only want brightness temperatures at Gaussian quadrature points (μ_i), we may simply calculate them using (5). But usually what we need is the brightness temperature at a given direction μ (53° for example). To calculate it, let's go back to the formal solutions of the radiative transfer equation:

$$I(\tau,+\mu) = I(\tau^*,+\mu)\exp[-(\tau^*-\tau)/\mu] + \int_{\tau}^{\tau^*} J(t,+\mu)\exp[-(t-\tau)/\mu]dt/\mu$$

$$I(\tau,-\mu) = I(0,-\mu)\exp(-\tau/\mu) + \int_0^{\tau} J(t,-\mu)\exp[-(\tau-t)/\mu]dt/\mu \tag{10}$$

where μ and –μ denote upward and downward directions. J is the source function including both emission and scattering. From (2), we have

$$J(\tau,\mu) = \tfrac{1}{2}\omega_0 \sum_{\ell=0}^{2n-1} A_\ell p_\ell(\mu) \sum_{j=-n}^{n} a_j p_\ell(\mu_j) I(\tau,\mu_i) + (1-\omega_0)(B_0 + B_1\tau) \tag{11}$$

The $I(\tau,\mu_i)$ in this equation are radiances from (5), i.e, radiances at Gaussian points. Substituting (5) into (11), we have

$$\begin{aligned} J(\tau,\mu) = & \sum_{j=1}^{n} (1+k_j\mu)(M_j+N_j)G_j(\mu)\exp(-k_j\tau) \\ & + \sum_{j=1}^{n} (1-k_j\mu)(M_j-N_j)H_j(\mu)\exp[-k_j(\tau^*-\tau)] \\ & + Z_0 + B_1\tau \end{aligned} \tag{12}$$

where

$$G_j(\mu) = \frac{\frac{1}{4}\omega_0}{1+k_j\mu} \sum_{\ell=0}^{2n-1} A_\ell p_\ell(\mu) \sum_{m=-n}^{n} a_m p_\ell(\mu_m) W_j(\mu_m)$$

$$H_j(\mu) = \frac{\frac{1}{4}\omega_0}{1-k_j\mu} \sum_{\ell=0}^{2n-1} A_\ell p_\ell(\mu) \sum_{m=-n}^{n} a_m p_\ell(\mu_m) W_{-j}(\mu_m)$$

$$Z_0 = \tfrac{1}{2}\omega_0 \sum_{\ell=0}^{2n-1} A_\ell p_\ell(\mu) \sum_{j=-n}^{n} a_j p_\ell(\mu_j) q(\mu_j) + (1-\omega_0)B_0$$

Substituting (12) in (10) and performing integration, it arrives:

$$
\begin{aligned}
I(\tau,+\mu) &= I(\tau^*,+\mu)\exp[-(\tau^*-\tau)/\mu] \\
&+ \sum_{j=1}^{2}(M_j+N_j)G_j(\mu)\{\exp(-k_j\tau)-\exp[-(k_j\tau^*+(\tau^*-\tau)/\mu)]\} \\
&+ \sum_{j=1}^{2}(M_j-N_j)H_j(\mu)\{\exp[-k_j(\tau^*-\tau)]-\exp[-(\tau^*-\tau)/\mu]\} \\
&+ Z_0\{1-\exp[-(\tau^*-\tau)/\mu]\} \\
&- B_1\{\tau^*\exp[-(\tau^*-\tau)/\mu]-\tau-\mu[1-\exp[-(\tau^*-\tau)/\mu]]\}
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
I(\tau,-\mu) &= I(0,-\mu)\exp(-\tau/\mu) \\
&+ \sum_{j=1}^{n}(M_j+N_j)G_j(-\mu)\left[\exp(-k_j\tau)-\exp(-\tau/\mu)\right] \\
&+ \sum_{j=1}^{n}(M_j-N_j)H_j(-\mu)\{\exp[-k_j(\tau^*-\tau)]-\exp[-(k_j\tau^*+\tau/\mu)] \\
&+ Z_0\left[1-\exp(-\tau/\mu)\right] \\
&+ B_1\left\{\tau-\mu\left[1-\exp(-\tau/\mu)\right]\right\}
\end{aligned}
\tag{14}
$$

In our problem, we are only interested in I(τ^*,-μ) and I(0,+μ), i.e, downward radiance at layer bottom and upward radiance at layer top. Note: downward radiance at top of layer *l* = downward radiance at bottom of layer *l*-1, and upward radiance at bottom of layer *l* = upward radiance at top of layer *l*+1. Again, layers are counted top-down. From (13) and (14), the two radiances are

$$
\begin{aligned}
I(0,+\mu) &= I(\tau^*,+\mu)\exp(-\tau^*/\mu) \\
&+ \sum_{j=1}^{n}(M_j+N_j)G_j(\mu)\{1-\exp[-(k_j\tau^*+\tau^*/\mu)]\} \\
&+ \sum_{j=1}^{n}(M_j-N_j)H_j(\mu)[\exp(-k_j\tau^*)-\exp(-\tau^*/\mu)] \\
&+ Z_0[1-\exp(-\tau^*/\mu)] \\
&- B_1\{\tau^*\exp(-\tau^*/\mu)-\mu[1-\exp(-\tau^*/\mu)]\}
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
I(\tau^*,-\mu) = I(0,-\mu)\exp(-\tau^*/\mu) \\
&+ \sum_{j=1}^{n}(M_j + N_j)G_j(-\mu)[\exp(-k_j\tau^*) - \exp(-\tau^*/\mu)] \\
&+ \sum_{j=1}^{n}(M_j - N_j)H_j(-\mu)\{1 - \exp[-(k_j\tau^* + \tau^*/\mu)]\} \\
&+ Z_0[1 - \exp(-\tau^*/\mu)] \\
&+ B_1\{\tau^* - \mu[1 - \exp(\tau^*/\mu)]\}
\end{aligned}
\tag{16}
$$

For satellite observations, what we need is $I^{(1)}(0,+\mu)$. Calculations may be performed in the following way:

Using top boundary condition $I^{(1)}(0,-\mu) = I_0(3\text{ K})$, calculate $I^{(1)}(\tau^*,-\mu)$ from (16). Then using continuity condition $I^{(2)}(0,-\mu) = I^{(1)}(\tau^*,-\mu)$, calculate $I^{(2)}(\tau^*,-\mu)$. Repeat this procedure till $I^{(L)}(\tau^*,-\mu)$ is obtained.

At lower boundary, for Fresnel surface, $I^{(L)}(\tau^*,+\mu) = \varepsilon(\mu)I_s + R(\mu)I^{(L)}(\tau^*,-\mu)$, where ε(μ)=1-R(μ).

For Lambertian surface, $I^{(L)}(\tau^*,+\mu) = \varepsilon I_s + 2R\sum_{k=1}^{n} I^{(L)}(\tau^*,-\mu_k)\mu_k a_k$, ε=1-R=const

This time, using (15) calculate upward radiance in a similar fashion as above till $I^{(1)}(0,+\mu)$ is obtained, which is the satellite-received radiance. Brightness temperature is then derived using Planck's function. All these calculations are done in gettb.f.